ADJUNCTION FOR THE GRAUERT–RIEMENSCHNEIDER CANONICAL SHEAF AND EXTENSION OF L^2-COHOMOLOGY CLASSES

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Abstract. In the present paper, we derive an adjunction formula for the Grauert-Riemenschneider canonical sheaf of a singular hypersurface V in a complex manifold M. This adjunction formula is used to study the problem of extending L^2-cohomology classes of $\overline{\partial}$-closed forms from the singular hypersurface V to the manifold M in the spirit of the Ohsawa-Takegoshi-Manivel extension theorem. We do that by showing that our formulation of the L^2-extension problem is invariant under bimeromorphic modifications, so that we can reduce the problem to the smooth case by use of an embedded resolution of V in M. The smooth case has recently been studied by Berndtsson.

Contents

1. Introduction

The L^2-theory for the $\overline{\partial}$-operator does not only play a central role in Complex Analysis itself, but has also led to fundamental advances in other fields as well, particularly in Algebraic Geometry. One very important complex analytic result with numerous applications to other areas is the celebrated Ohsawa-Takegoshi L^2-extension theorem [OT], which has been generalized later by many authors. In

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essence, the statement says that holomorphic L^2-sections of a suitably positive line bundle over a subvariety Y of a complex manifold X extend to holomorphic L^2-sections over the whole of X, and that there exist good L^2-estimates for this extension procedure. The positivity condition can be understood in terms of the canonical bundle of X and the normal bundle of Y in X.

Some of the important results that use the Ohsawa-Takegoshi extension theorem are e.g. Siu's theorems on the analyticity of the sublevel sets of Lelong numbers [S1] and the invariance of plurigenera [S3], approximation of closed positive currents by analytic cycles, subadditivity of multiplier ideal sheaves and Fujita's approximate Zariski decomposition (for all this we refer to Demailly [D2]). Quite recently, Błocki has used a version of the Ohsawa-Takegoshi extension theorem with optimal constants to prove the Suita conjecture (see [B3]).

An essential development was Manivel's geometrically motivated generalization of the L^2-extension theorem to the framework of vector bundles and $\overline{\partial}$-closed forms of higher degree [M]. A simplified proof of Manivel's results can be found in [D2], where Demailly also points out a difficulty in the proof of the smoothness of $\overline{\partial}$-closed extensions for forms of higher degree. Following the ideas of Demailly, Koziarz recently bypassed this problem by considering the extension of $\overline{\partial}$-cohomology classes instead of the extension of individual forms (see [K2]). This is a natural setting for many kinds of extension problems. Koziarz's method is inspired by Siu's idea to represent cohomology classes by Čech cocycles [S2].

Recently, Berndtsson improved Koziarz's result considerably by showing that one can actually get an L^2-extension theorem for individual smooth $\overline{\partial}$-closed $(0,q)$-forms, $q \geq 0$, with an absolute constant for the extension under quite weak positivity assumptions (see [B2]). Let us explain Berndtsson's theorem more precisely.

Let X be a compact Kähler manifold of dimension n with Kähler form ω and L a holomorphic line bundle over X. Let Δ be a smooth divisor in X, given as the zero set $\Delta = s^{-1}(0)$ of a holomorphic section s of a line bundle S. The latter is usually obtained as follows: Let $S = [\Delta]$ be the line bundle associated to $\mathcal{O}(\Delta)$, i.e. to the sheaf of meromorphic functions with at most a single pole along Δ, and let s be the section of S induced by the holomorphic function $f \equiv 1$. Then $\Delta = s^{-1}(0)$. We will freely allow ourselves to identify meromorphic functions in $\mathcal{O}(\Delta)$ with holomorphic sections of the line bundle $[\Delta]$.

Note that $[\Delta]|_\Delta \cong N_\Delta$, where N_Δ is the normal bundle of Δ in X. By a slight abuse of notation, we will sometimes just call $[\Delta]$ the normal bundle of Δ.

Generalizing his approach from [B1], Berndtsson proved the following extension theorem under rather weak positivity assumptions:

Theorem 1.1. (Berndtsson [B2]**)** *Assume that ϕ is a smooth metric on L and that ψ is a smooth metric on S such that*

$$i\partial\overline{\partial}\phi \wedge \omega^q \geq \epsilon i\partial\overline{\partial}\psi \wedge \omega^q$$

and

$$i\partial\overline{\partial}\phi \wedge \omega^q \geq 0.$$

Assume moreover the normalizing inequality

$$\log(|s|^2 e^{-2\psi}) \leq -1/\epsilon.$$

Let u be a smooth $\overline{\partial}$-closed $(n-1,q)$-form with values in L over Δ. Then there is a $\overline{\partial}$-closed (n,q)-form U with values in $S+L$ over X such that

$$U = ds \wedge u \tag{1}$$

on Δ and

$$\int_X |U|^2 e^{-2\phi-2\psi} dV_X \leq C_0 \int_\Delta |u|^2 e^{-2\phi} dV_\Delta,$$

where $C_0 > 0$ is an absolute constant. The norms and the volume forms are defined by the Kähler form ω.

In contrast to the original Ohsawa-Takegoshi extension [OT] (which only deals with the case $q=0$), the method of Berndtsson [B2] is not applicable to a singular divisor Δ. Also Koziarz's approach [K2] covers only the case of a smooth divisor, whereas Manivel [M] and Demailly [D2] consider the singular setting but do not achieve smooth extension for $q>0$. In the present paper, we study some aspects of the generalization of Theorem 1.1 to the situation of a singular divisor Δ.

We must specify what kind of objects we wish to extend and how the relation of the original object to the extension should look like (in the spirit of equation (1)). We consider the following situation: let M be a compact complex manifold and V a hypersurface in M. When V is smooth, then the relation (1) is induced by the adjunction formula which sits at the core of the classical extension theorems:

$$K_V = \big(K_M \otimes [V]\big)\big|_V = K_M|_V \otimes N_V, \tag{2}$$

where K_V and K_M denote the canonical bundles of V and M, respectively, $[V]$ is the line bundle associated to $\mathcal{O}(V)$, and N_V is the normal bundle of V in M.

In the language of sheaves, (2) can be understood as the short exact sequence

$$0 \to \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \xrightarrow{\Psi} \iota_* \mathcal{K}_V \to 0, \tag{3}$$

where $\mathcal{K}_V$ and $\mathcal{K}_M$ are the canonical sheaves of V and M, respectively, and $\iota_*\mathcal{K}_V$ is the trivial extension of $\mathcal{K}_V$ to M. Ψ is the mapping defined by the local equation

$$\eta = \frac{df}{f} \wedge \Psi(\eta) \tag{4}$$

in $\mathcal{K}_M \otimes \mathcal{O}(V)$ over V where f is any local defining function for the divisor V. More concretely, if z are local holomorphic coordinates on M such that $V = \{z_1 = 0\}$, then $\Psi(g dz_1 \wedge \cdots \wedge dz_n / z_1) = g|_V dz_2 \wedge \cdots \wedge dz_n$.

When V is a singular hypersurface, the adjunction formula (3) remains valid if we replace $\mathcal{K}_V$ by Grothendieck's dualizing sheaf $\omega_V = \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{O}_V, \mathcal{K}_M)$, sometimes also called the Barlet sheaf (see e.g. [PR], §5.3). However, this would not be in accordance with the classical Ohsawa-Takegoshi-Manivel extension theorem as sections of ω_V are in general not square-integrable $(n-1)$-forms on the regular part of V, but may have poles of higher order at the singular set $\operatorname{Sing} V$ of V. In contrast, all the classical extension theorems (see [OT], [M], [D2]) are about extension of holomorphic L^2-forms from a possibly singular divisor.

Thus, we propose to replace the canonical sheaf of V in the singular case by the Grauert-Riemenschneider canonical sheaf of holomorphic square-integrable $(n-1)$-forms on (the regular part of) V. We denote that sheaf again by $\mathcal{K}_V$. Note that $\omega_V \cong i_*(\omega_{V \setminus \operatorname{Sing} V})$ for the natural inclusion $i : V \setminus \operatorname{Sing} V \hookrightarrow V$ if V is normal (see [GR]). So, if V is normal, then $\mathcal{K}_V \subset \omega_V$, and this inclusion is strict in general. In

contrast to Grothendieck's dualizing sheaf, which is important in the context of Serre duality, the canonical sheaf of Grauert-Riemenschneider is particularly important in the context of modifications of complex spaces as the L^2-property of n-forms is insensitive to such operations.

Our first main result of the present paper is the adjunction formula for the Grauert-Riemenschneider canonical sheaf, cf. Section 3.2.

Theorem 1.2. *Let V be a (possibly singular) hypersurface in a Hermitian complex manifold M. Then there exists a unique multiplier ideal sheaf $\mathcal{J}(V)$ such that there is a natural short exact sequence*

$$0 \to \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \xrightarrow{\Psi} \iota_* \mathcal{K}_V \to 0, \tag{5}$$

where $\mathcal{K}_M$ is the usual canonical sheaf of M, $\iota_*\mathcal{K}_V$ is the trivial extension of the Grauert-Riemenschneider canonical sheaf $\mathcal{K}_V$ of V to M, and Ψ is the adjunction map from (4). *The zero set of $\mathcal{J}(V)$ is contained in* $\operatorname{Sing} V$.

As a consequence, we can deduce:

Theorem 1.3. *The natural inclusion $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V)$ induces a natural isomorphism*

$$\mathcal{K}_V \cong \omega_V \otimes \mathcal{J}(V)|_V.$$

In particular, there is a natural inclusion of the Grauert-Riemenschneider canonical sheaf into Grothendieck's dualizing sheaf, $\mathcal{K}_V \subset \omega_V$.

Recall that for normal V, the inclusion $\mathcal{K}_V \subset \omega_V$ is due to Grauert-Riemenschneider (see [GR]). It is clearly interesting to know under which circumstances $\mathcal{K}_V = \omega_V$, i.e. $\mathcal{J}(V) = \mathcal{O}_M$. If V is normal, then this is the case precisely if V has only canonical singularities as appearing in the minimal model program:

Theorem 1.4. *Let V be a normal hypersuface in M. Then $\mathcal{K}_V = \omega_V$, i.e. $\mathcal{J}(V) = \mathcal{O}_M$, exactly if V has only canonical singularities.*

Note that Theorem 1.2 implies extension of (flabby) cohomology classes by use of the long exact cohomology sequence in the sense that the natural mapping $H^q\big(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big) \longrightarrow H^q(V, \mathcal{K}_V)$ is surjective if $H^{q+1}(M, \mathcal{K}_M) = 0$.

In our adjunction formula (5), we have replaced the sheaf $\mathcal{O}(V)$ by the sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ which we will call the **adjunction sheaf** of V in M. Let us now discuss L^2-extension of $\overline{\partial}$-closed forms in the spirit of (5). We have to face the problem that $\mathcal{O}(V) \otimes \mathcal{J}(V)$ is in general not locally free. Whereas $\mathcal{O}(V)$ is associated to the line bundle $[V]$ so that one can usually work with a Hermitian line bundle $([V], e^{-\psi})$, we need to introduce a concept generalizing that. If E is an effective divisor in M' there is an associated canonical singular metric $e^{-2\varphi}$ on $[E]$. If E is locally defined by f_α, then φ_E is locally given as $\log |f_\alpha|$.

Definition 1.5. *Let $e^{-2\psi}$ be a singular metric on the normal bundle $[V]$ of the hypersurface V in M. We say that $e^{-2\psi}$ is a* **smooth metric** *on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ if there exists an embedded resolution of singularities with only normal crossings $\pi : (V', M') \to (V, M)$ such that $e^{-2\pi^*\psi + 2\varphi_E}$ is a smooth metric on the normal bundle $[V']$ of V' in M'.*

We believe that this concept is an appropriate generalization of the usual normal bundle with a smooth metric, to be used in the context of L^2-extension of $\overline{\partial}$-closed forms from a singular divisor. In fact, we can prove some natural extension results.

Any smooth metric on $[V']$ induces a singular metric on $[V]$ which is a smooth metric on $\mathcal{O}(V)\otimes\mathcal{J}(V)$ in the sense of Definition 1.5 if $\pi:(V',M')\to(V,M)$ is any embedded resolution of V in M with only normal crossings (see Theorem 4.1).

We need to explain shortly what we understand by L^2-cohomology of a (singular) Hermitian complex space. For any Hermitian complex space X and a Hermitian line bundle $F\to X$ (with possibly singular metric), we denote by $\mathcal{C}_X^{p,q}(F)$ the sheaf of germs of (p,q)-forms g on the regular part of X (with values in F) which are square-integrable up to the singular set and which are in the domain of the $\overline{\partial}$-operator in the sense of distributions, $\overline{\partial}_w$, meaning that $\overline{\partial}_w g$ is again square-integrable up to the singular set (see Section 2 for details). For an open set $U\subset X$, we define the L^2-cohomology

$$H^{p,q}_{(2)}(U^*,F):=H^q\big(\Gamma(U,(\mathcal{C}_X^{p,*}(F),\overline{\partial}_w))\big),$$

i.e. consider the cohomology of the complex $(\mathcal{C}_X^{p,*}(F),\overline{\partial}_w)$. Here and in the following, let $U^*=\operatorname{Reg} U$ denote the regular part of U.

We can now exemplify how our definition of a smooth metric on $\mathcal{O}(V)\otimes\mathcal{J}(V)$ connects the adjunction sheaf to L^2-cohomology (Theorem 4.2 and Theorem 4.4):

Theorem 1.6. *Let M be a compact complex manifold of dimension n, and let $[V]^{sing}:=([V],e^{-\psi})$ be the normal bundle of a hypersurface V in M carrying a singular Hermitian metric which is a smooth metric on the adjunction sheaf $\mathcal{O}(V)\otimes\mathcal{J}(V)$ in the sense of Definition 1.5. Then*

$$\mathcal{O}(V)\otimes\mathcal{J}(V)\cong\ker\overline{\partial}_w\subset\mathcal{C}_M^{0,0}([V]^{sing}).$$

If $F\to M$ is any Hermitian line bundle (with smooth metric), then the complex $\big(\mathcal{C}_M^{n,*}(F\otimes[V]^{sing}),\overline{\partial}_w\big)$ is a fine resolution for $\mathcal{K}_M(F)\otimes\mathcal{O}(V)\otimes\mathcal{J}(V)$, where we denote by $\mathcal{K}_M(F)$ the sheaf of holomorphic n-forms with values in F. Thus

$$H^q\big(M,\mathcal{K}_M(F)\otimes\mathcal{O}(V)\otimes\mathcal{J}(V)\big)\cong H^{n,q}_{(2)}\big(M,F\otimes[V]^{sing}\big).$$

The proof of Theorem 1.6 relies on the fact that the metric $e^{-\psi}$ behaves locally like $e^{-\varphi}$ where φ is a plurisubharmonic defining function for the multiplier ideal sheaf $\mathcal{J}(V)$.

Using Theorem 1.6, we will deduce that the adjunction map Ψ from (5) in Theorem 1.2 induces a natural map Ψ_V on the level of L^2-cohomology,

$$\Psi_V:H^{n,q}_{(2)}(M,F\otimes[V]^{sing})\to H^{n-1,q}_{(2)}(V^*,F),$$

and the extension problem for L^2-cohomology classes amounts to studying the question whether Ψ_V is surjective.

Our second main result, Theorem 4.6, shows that this L^2-extension problem is invariant under bimeromorphic modifications. This allows to reduce the problem to the case of a smooth divisor where we can apply Berndtsson's Theorem 1.1. Let $\pi:(V',M')\to(V,M)$ be an embedded resolution of V in M such that $\pi^*e^{-\psi}$ induces a smooth metric on the normal bundle $[V']$ of V' in M'. Then there exists a natural commutative diagram

$$\begin{tikzcd}
H^{n,q}_{(2)}(M,F\otimes[V]^{sing}) \arrow[r, "\Psi_V"] \arrow[d, "\pi^*", "\cong"'] & H^{n-1,q}_{(2)}(V^*,F) \arrow[d, "\pi^*", "\cong"'] \\
H^{n,q}(M',\pi^*F\otimes[V']) \arrow[r, "\Psi_{V'}"] & H^{n-1,q}_{(2)}(V',\pi^*F),
\end{tikzcd} \tag{6}$$

where $\Psi_{V'}$ is the adjunction map for the smooth divisor V' in M'. The vertical maps π^* are induced by pull-back of forms under π and both are isomorphisms.

By an easy application of Berndtsson's Theorem 1.1, one can now deduce the following theorem, see Section 4.4.

Theorem 1.7. *Let $e^{-\phi}$ be the smooth metric of the Hermitian line bundle $F \to M$, and assume furthermore that M is a Kähler manifold with Kähler metric ω. Let $0 \leq q \leq n-1$. Then the adjunction map Ψ_V in (6) is surjective if the following condition is satisfied:*

There exists an $\epsilon > 0$ such that $i\partial\overline{\partial}\phi \wedge \omega^q \geq \epsilon i\partial\overline{\partial}\psi \wedge \omega^q$ and $i\partial\overline{\partial}\phi \wedge \omega^q \geq 0$, where $e^{-\psi}$ is the singular metric on $[V]$ which is smooth on $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

Another condition to ensure surjectivity of the adjunction map Ψ_V in (6) is $H^{n,q+1}(M,F) = 0$. That can be seen by applying the long exact cohomology sequence to the adjunction formula (5) in Theorem 1.2. Ψ_V is surjective e.g. if $F \to M$ is a positive line bundle by Kodaira's vanishing theorem.

We also show that L^2-cohomology classes in $H^{n,q}_{(2)}(M, F \otimes [V]^{sing})$ have smooth representatives in $\Gamma\big(M, \mathcal{C}^\infty_{n,q}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big)$. That yields:

Corollary 1.8. *Assume that the map Ψ_V in (6) is surjective. Let $u \in \Gamma\big(V, \mathcal{C}^{n-1,q}_V(F)\big)$ be a $\overline{\partial}$-closed L^2-form of degree $(n-1,q)$ on the singular hypersurface V.*

If $q \geq 1$ then there exists an L^2-form $g \in \Gamma\big(V, \mathcal{C}^{n-1,q-1}_V(F)\big)$ and a $\overline{\partial}$-closed section $U \in \Gamma\big(M, \mathcal{C}^\infty_{n,q}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big)$, i.e. a smooth (n,q)-form with values in $F \otimes [V]$ with some extra vanishing according to $\mathcal{J}(V)$, such that $U = \frac{df}{f} \wedge (u - \overline{\partial} g)$ where f is any local defining function for the hypersurface V.

For $q = 0$ the statement holds without g.

We remark that one can deduce some L^2-estimates for our extension by use of Berndtsson's theorem, but forgo this topic here as these estimates would depend on the resolution of singularities that is used. Anyway, it is easy to see (and a nice feature) that one can tensor F in such statements with another semi-positive line bundle $F' \to M$ without changing the estimate.

2. The canonical sheaf of Grauert–Riemenschneider $\mathcal{K}_X$

2.1. **Bimeromorphically invariant L^2-cohomology.**

Our intention is to study the L^2-extension of $\overline{\partial}$-cohomology classes by reducing the problem to a smooth setting. It turns out that this is possible if we consider the L^2-cohomology of (n,q)-forms with respect to the $\overline{\partial}$-operator in the sense of distributions on the regular part of a singular Hermitian space. In fact, these cohomology groups are bimeromorphically invariant:

Theorem 2.1. ([R2], **Theorem 4.1)** *Let X be a Hermitian compact complex space of pure dimension n, $\pi : M \to X$ any resolution of singularities and $0 \leq q \leq n$. Then there is a canonical isomorphism*

$$\pi^* : H^{n,q}_{(2)}(X \setminus \operatorname{Sing} X) \xrightarrow{\cong} H^{n,q}_{(2)}(M) \tag{7}$$

induced by pull-back of forms under π, where we denote by $H^{n,q}_{(2)}$ the L^2-cohomology with respect to the $\overline{\partial}$-operator in the sense of distributions.

Let $L \to M$ be a Hermitian holomorphic line bundle and $\pi_* L \to X \setminus \operatorname{Sing} X$ its direct image bundle, i.e. $\pi_* L = (\pi|_{M\setminus E}^{-1})^* L$, where E is the exceptional set of the

resolution $\pi : M \to X$. Assume that L is locally semi-positive with respect to the base space X, i.e. that each point $x \in X$ has a neighborhood U_x such that L is semi-positive on $\pi^{-1}(U_x)$. Then (7) remains valid for forms with values in L, i.e. there is a canonical isomorphism

$$\pi^* : H^{n,q}_{(2)}(X \setminus \operatorname{Sing} X, \pi_* L) \xrightarrow{\cong} H^{n,q}_{(2)}(M, L) \tag{8}$$

induced by pull-back of forms under π.

This is the unchallenged prototype to illustrate the general philosophy to use a resolution of singularities to obtain a regular model for the L^2-cohomology on a singular space. One can deduce immediately that the groups $H^{n,q}_{(2)}(X \setminus \operatorname{Sing} X)$ are of finite dimension and that the $\overline{\partial}$-operator in the L^2-sense of distributions has closed range for (n,q)-forms on $X \setminus \operatorname{Sing} X$ (this follows from Theorem 2.1 directly by an argument in [HL], Appendix 2.4).

The main tools for the proof of Theorem 2.1 are Hironaka's resolution of singularities [H2], the canonical sheaf of Grauert–Riemenschneider [GR], Takegoshi's relative vanishing theorem for canonical sheaves [T], and a local vanishing result which is based on results of Demailly [D1], Donelly–Fefferman [DF], Ohsawa [O] and which is finally due to Pardon-Stern [PS]. Pardon and Stern proved the first statement (7) of Theorem 2.1 for projective varieties in [PS].

We need to recall the proof of Theorem 2.1. Let us first define the canonical sheaf $\mathcal{K}_X$ of Grauert and Riemenschneider which is the key object that we have to study. It is the sheaf of germs of square-integrable n-forms which are holomorphic with respect to the localized version of the $\overline{\partial}$-operator in the sense of distributions which we will denote by $\overline{\partial}_w$ in the following.

2.2. **The weak $\overline{\partial}$-operator $\overline{\partial}_w$ and its L^2-complex.** We recall some of the essential constructions from [R2]. Let (X,h) always be a (singular) Hermitian complex space of pure dimension n, $F \to X \setminus \operatorname{Sing} X$ a Hermitian holomorphic line bundle, and $U \subset X$ an open subset. On a singular space, it is most fruitful to consider forms that are square-integrable up to the singular set. Hence, we will use the following concept of locally square-integrable forms with values in F:

$$L^{2,loc}_{p,q}(U,F) := \{f \in L^{2,loc}_{p,q}(U \setminus \operatorname{Sing} X, F) : f|_K \in L^2_{p,q}(K \setminus \operatorname{Sing} X, F)\ \forall K \subset\subset U\}.$$

It is easy to check that the presheaves given as

$$\mathcal{L}^{p,q}(U,F) := L^{2,loc}_{p,q}(U,F)$$

are already sheaves $\mathcal{L}^{p,q}(F) \to X$. On $L^{2,loc}_{p,q}(U,F)$, we denote by

$$\overline{\partial}_w(U) : L^{2,loc}_{p,q}(U,F) \to L^{2,loc}_{p,q+1}(U,F)$$

the $\overline{\partial}$-operator in the sense of distributions on $U \setminus \operatorname{Sing} X$ which is closed and densely defined. When there is no danger of confusion, we will simply write $\overline{\partial}_w$ for $\overline{\partial}_w(U)$. The subscript refers to $\overline{\partial}_w$ as an operator in a weak sense. Since $\overline{\partial}_w$ is a local operator, i.e. $\overline{\partial}_w(U)|_V = \overline{\partial}_w(V)$ for open sets $V \subset U$, we can define the presheaves of germs of forms in the domain of $\overline{\partial}_w$,

$$\mathcal{C}^{p,q}(F) := \mathcal{L}^{p,q}(F) \cap \overline{\partial}_w^{-1} \mathcal{L}^{p,q+1}(F),$$

given by

$$\mathcal{C}^{p,q}(U,F) = \mathcal{L}^{p,q}(U,F) \cap \operatorname{Dom} \overline{\partial}_w(U).$$

These are actually already sheaves because the following is also clear: If $U = \bigcup U_\mu$ is a union of open sets, $f_\mu = f|_{U_\mu}$ and $f_\mu \in \operatorname{Dom} \overline{\partial}_w(U_\mu)$, then

$$f \in \operatorname{Dom} \overline{\partial}_w(U) \quad \text{ and } \quad \big(\overline{\partial}_w(U) f\big)|_{U_\mu} = \overline{\partial}_w(U_\mu) f_\mu.$$

Moreover, it is easy to see that the sheaves $\mathcal{C}^{p,q}(F)$ admit partitions of unity, and so we obtain fine sequences

$$\mathcal{C}^{p,0}(F) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{p,1}(F) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{p,2}(F) \xrightarrow{\overline{\partial}_w} \dots \tag{9}$$

We use simply $\mathcal{C}^{p,q}$ to denote the sheaves of forms with values in the trivial line bundle. We will see later, when we deal with resolution of singularities, that

$$\mathcal{K}_X := \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}$$

is just the canonical sheaf of Grauert and Riemenschneider because the L^2-property of $(n,0)$-forms remains invariant under modifications of the metric.

The $L^{2,loc}$-Dolbeault cohomology for forms with values in F with respect to the $\overline{\partial}_w$-operator on an open set $U \subset X$ is by definition the cohomology of the complex (9) after taking global sections over U; this is denoted by $H^q(\Gamma(U, \mathcal{C}^{p,*}(F)))$. The cohomology with compact support is $H^q(\Gamma_{cpt}(U, \mathcal{C}^{p,*}(F)))$. Note that this is the cohomology of forms with compact support in U, not with compact support in $U \setminus \operatorname{Sing} X$.

It is clearly interesting to study whether the sequence (9) is exact, which is well-known to be the case in regular points of X. In singular points, the situation is quite complicated for forms of arbitrary degree and not completely understood. However, the $\overline{\partial}_w$-equation is locally solvable in the L^2-sense at arbitrary singularities for forms of degree (n,q), $q > 0$, with values in a Hermitian holomorphic line bundle which is locally semi-positive with respect to X:

Theorem 2.2. ([R2], **Lemma 4.4**) *Let X be a Hermitian complex space of pure dimension n, and $F \to X \setminus \operatorname{Sing} X$ a Hermitian holomorphic line bundle which is locally semi-positive on X, i.e. each point $x \in X$ has a neighborhood U_x such that F is semi-positive on $U_x \setminus \operatorname{Sing} X$.*[1]

Then

$$0 \to \mathcal{K}_X(F) \hookrightarrow \mathcal{C}^{n,0}(F) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,1}(F) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,2}(F) \xrightarrow{\overline{\partial}_w} \dots \longrightarrow \mathcal{C}^{n,n}(F) \to 0 \tag{10}$$

is a fine resolution, where we set $\mathcal{K}_X(F) := \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}(F)$.

For an open set $U \subset X$, it follows that

$$H^q(U, \mathcal{K}_X(F)) \cong H^q(\Gamma(U, \mathcal{C}^{n,*}(F))) \ , \ H^q_{cpt}(U, \mathcal{K}_X(F)) \cong H^q(\Gamma_{cpt}(U, \mathcal{C}^{n,*}(F))).$$

For forms with values in the trivial bundle, the statement of Theorem 2.2 is due to Pardon and Stern (see [PS], Proposition 2.1). If X has only isolated singularities, Fornæss–Øvrelid–Vassiliadou showed that the $\overline{\partial}_w$-equation is locally solvable in the L^2-sense for forms of degree (p,q) with $p+q > n$ (see [FOV], Theorem 1.2).

The main idea for the proof of Theorem 2.2 is as follows. Locally, one can approximate the incomplete metric on $X \setminus \operatorname{Sing} X$ by a sequence of complete Kähler metrics for which one already knows the local vanishing result by a theorem of Donelly and Fefferman [DF]. One obtains a sequence of solutions with a uniform L^2-bound on compact subsets of $X \setminus \operatorname{Sing} X$. By taking the weak limit, we get a solution with

[1] Note that this condition is trivially fulfilled if F extends to a line bundle over X. So, the theorem applies to forms with values in the trivial line bundle over X, i.e. the complex $(\mathcal{C}^{n,*}, \overline{\partial}_w)$ is an exact resolution of the Grauert-Riemenschneider canonical sheaf $\mathcal{K}_X$.

an L^2-bound in the original metric. This strategy was used before by Ohsawa in the case when X has only isolated singularities [O], but also appears in an earlier paper of Demailly [D1]. For the details, we refer to [PS], Proposition 2.1, or to Sections 3 and 4 in [R2], where also the generalization to (n,q)-forms with values in the semi-positive line bundle F can be found.

A similar local L^2-vanishing result for (n,q)-forms on positive closed currents of bidimension (n,n) has been proved by Berndtsson and Sibony [BS].

2.3. **Resolution of $(X,\mathcal{K}_X)$ and its L^2-cohomology.** Let $\pi : M \to X$ be a resolution of singularities (which exists due to Hironaka [H2]), i.e. a proper holomorphic surjection such that

$$\pi|_{M\setminus E} : M \setminus E \to X \setminus \operatorname{Sing} X$$

is biholomorphic, where $E = |\pi^{-1}(\operatorname{Sing} X)|$ is the exceptional set. We may assume that E is a divisor with only normal crossings, i.e. the irreducible components of E are regular and meet complex transversely, but we do not need that for the moment. Let $Z := \pi^{-1}(\operatorname{Sing} X)$ be the unreduced exceptional divisor. For the topic of desingularization, we refer to [AHL], [BM] and [H1].

Let $\gamma := \pi^* h$ be the pullback of the Hermitian metric h of X to M; γ is positive semidefinite (a pseudo-metric) with degeneracy locus E. Notice that h is smooth on X in the sense that if $X \hookrightarrow \mathbb{C}^N$ is a local embedding then h is the restriction to $\operatorname{Reg} X$ of a smooth ambient metric. Hence γ is smooth on M.

We give M the structure of a Hermitian manifold with a freely chosen (positive definite) metric σ. Then $\gamma \lesssim \sigma$ and $\gamma \sim \sigma$ on compact subsets of $M \setminus E$. For an open set $U \subset M$, we denote by $L^{p,q}_\gamma(U)$ and $L^{p,q}_\sigma(U)$ the spaces of square-integrable (p,q)-forms with respect to the (pseudo-)metrics γ and σ, respectively.

Since σ is positive definite and γ is positive semi-definite, there exists a continuous function $g \in C^0(M,\mathbb{R})$ such that $dV_\gamma = g^2 dV_\sigma$. This yields $|g||\omega|_\gamma = |\omega|_\sigma$ if ω is an $(n,0)$-form, and $|\omega|_\sigma \lesssim_U |g||\omega|_\gamma$ on $U \subset\subset M$ if ω is a (n,q)-form, $0 \le q \le n$.[2] So, for an (n,q) form ω on $U \subset\subset M$:

$$\int_U |\omega|^2_\sigma dV_\sigma \lesssim_U \int_U g^2 |\omega|^2_\gamma g^{-2} dV_\gamma = \int_U |\omega|^2_\gamma dV_\gamma.$$

Conversely, $|g||\eta|_\gamma \lesssim_U |\eta|_\sigma$ on $U \subset\subset M$ if η is a $(0,q)$-form, $0 \le q \le n$.[3] So, for a $(0,q)$ form η on $U \subset\subset M$:

$$\int_U |\eta|^2_\gamma dV_\gamma \lesssim_U \int_U g^{-2} |\eta|^2_\sigma g^2 dV_\sigma = \int_U |\eta|^2_\sigma dV_\sigma.$$

For open sets $U \subset\subset M$ and all $0 \le q \le n$, we conclude the relations

$$\begin{aligned} L^{n,q}_\gamma(U) &\subset L^{n,q}_\sigma(U), \\ L^{0,q}_\sigma(U) &\subset L^{0,q}_\gamma(U). \end{aligned}$$

If $L \to M$ is a Hermitian holomorphic line bundle over M, we have:

$$L^{n,q}_\gamma(U,L) \subset L^{n,q}_\sigma(U,L), \tag{11}$$

$$L^{0,q}_\sigma(U,L) \subset L^{0,q}_\gamma(U,L). \tag{12}$$

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[2] This statement means that $|\omega|_\sigma/|\omega|_\gamma$ is locally bounded on M for (n,q)-forms.

[3] For $(0,q)$-forms, $|\omega|_\gamma/|\omega|_\sigma$ is locally bounded.

For an open set $\Omega \subset X$, $\Omega^* = \Omega \setminus \operatorname{Sing} X$, $\widetilde{\Omega} := \pi^{-1}(\Omega)$, pullback of forms under π gives the isometry

$$\pi^* : L^2_{p,q}(\Omega^*) \longrightarrow L^{p,q}_{\gamma}(\widetilde{\Omega} \setminus E) \cong L^{p,q}_{\gamma}(\widetilde{\Omega}),$$

where the last identification is by trivial extension of forms over the thin exceptional set E. If $\pi_* L \to X \setminus \operatorname{Sing} X$ is the direct image, i.e. $\pi_* L = (\pi|^{-1}_{M\setminus E})^* L$, then π gives analogously the isometry

$$\pi^* : L^2_{p,q}(\Omega^*, \pi_* L) \longrightarrow L^{p,q}_{\gamma}(\widetilde{\Omega} \setminus E, L) \cong L^{p,q}_{\gamma}(\widetilde{\Omega}, L). \tag{13}$$

Combining (11) with (13), we see that π^* maps

$$\pi^* : L^2_{n,q}(\Omega^*, \pi_* L) \to L^{n,q}_{\sigma}(\pi^{-1}(\Omega), L) \tag{14}$$

continuously if $\Omega \subset\subset X$ is a relatively compact open set. We shall now show how (14) induces the map[4]

$$\pi^* : H^{n,q}_{(2)}(X \setminus \operatorname{Sing} X, \pi_* L) \to H^{n,q}_{(2)}(M, L) \tag{15}$$

from Theorem 2.1 (where X is compact).

It makes sense to explain that from a slightly more general point of view. For that, we need a suitable realization of the L^2-cohomology on M. Let $\mathcal{L}^{p,q}_{\sigma}(L)$ be the sheaves of germs of forms on M which are locally in $L^{p,q}_{\sigma}(L)$, and we denote again by $\overline{\partial}_w$ the $\overline{\partial}$-operator in the sense of distributions on such forms because there is no danger of confusion in what follows. We can simply use the definitions from Section 2.2 with the choice $X = M$ and $\operatorname{Sing} X = \emptyset$. Again, we denote the sheaves of germs in the domain of $\overline{\partial}_w$ by

$$\mathcal{C}^{p,q}_{\sigma}(L) := \mathcal{L}^{p,q}_{\sigma}(L) \cap \overline{\partial}_w^{-1} \mathcal{L}^{p,q+1}_{\sigma}(L)$$

in the sense that

$$\mathcal{C}^{p,q}_{\sigma}(U, L) = \mathcal{L}^{p,q}_{\sigma}(U, L) \cap \operatorname{Dom} \overline{\partial}_w(U).$$

It is well-known that

$$\mathcal{K}_M(L) := \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}_{\sigma}(L)$$

is the usual canonical sheaf on M if L is the trivial line bundle, and that

$$0 \to \mathcal{K}_M(L) \hookrightarrow \mathcal{C}^{n,0}_{\sigma}(L) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,1}_{\sigma}(L) \xrightarrow{\overline{\partial}_w} \mathcal{C}^{n,2}_{\sigma}(L) \longrightarrow \ldots \tag{16}$$

is a fine resolution so that

$$H^q(U, \mathcal{K}_M(L)) \cong H^q(\Gamma(U, \mathcal{C}^{n,*}_{\sigma}(L))) \ , \ H^q_{cpt}(U, \mathcal{K}_M(L)) \cong H^q(\Gamma_{cpt}(U, \mathcal{C}^{n,*}_{\sigma}(L)))$$

on open sets $U \subset M$.

Now we can use (14) to see that π^* induces a morphism of complexes

$$\pi^* : (\mathcal{C}^{n,*}(\pi_* L), \overline{\partial}_w) \to (\pi_*(\mathcal{C}^{n,*}_{\sigma}(L)), \pi_* \overline{\partial}_w). \tag{17}$$

Let $\Omega \subset X$ be an open set and let $f \in \mathcal{C}^{n,q}(\Omega, \pi_* L)$, $g \in \mathcal{C}^{n,q+1}(\Omega, \pi_* L)$ such that $\overline{\partial}_w f = g$. By (14), it follows that $\pi^* f \in \mathcal{L}^{n,q}_{\sigma}(\pi^{-1}(\Omega), L)$ and $\pi^* g \in \mathcal{L}^{n,q+1}_{\sigma}(\pi^{-1}(\Omega), L)$ so that $\overline{\partial}_w \pi^* f = \pi^* g$ on $\pi^{-1}(\Omega) \setminus E$. But then the L^2-extension theorem [R1], Theorem 3.2, tells us that $\overline{\partial}_w \pi^* f = \pi^* g$ on $\pi^{-1}(\Omega)$. So $\pi^* f \in \mathcal{C}^{n,q}_{\sigma}(\pi^{-1}(\Omega), L)$, $\pi^* g \in \mathcal{C}^{n,q+1}_{\sigma}(\pi^{-1}(\Omega), L)$ and (17) is in fact a morphism of complexes. Including

[4]To simplify notation we somewhat abusively also denote the induced map by π^*.

$\mathcal{K}_X(\pi_* L) = \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}(\pi_* L)$ and $\mathcal{K}_M(L) = \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}_\sigma(L)$, we obtain the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathcal{K}_X(\pi_*L) & \longrightarrow & \mathcal{C}^{n,0}(\pi_*L) & \xrightarrow{\overline{\partial}_w} & \mathcal{C}^{n,1}(\pi_*L) & \xrightarrow{\overline{\partial}_w} & \mathcal{C}^{n,2}(\pi_*L) & \longrightarrow & \ldots \\
 & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} & & \\
0 & \longrightarrow & \pi_*(\mathcal{K}_M(L)) & \longrightarrow & \pi_*(\mathcal{C}^{n,0}_\sigma(L)) & \xrightarrow{\pi_*\overline{\partial}_w} & \pi_*(\mathcal{C}^{n,1}_\sigma(L)) & \xrightarrow{\pi_*\overline{\partial}_w} & \pi_*(\mathcal{C}^{n,2}_\sigma(L)) & \longrightarrow & \ldots
\end{array}$$

Note that the upper line is exact by Theorem 2.2 if $\pi_* L$ is locally semi-positive on X, e.g. when L is the trivial bundle.

It follows from commutativity of the diagram that π^* induces a morphism on the cohomology of the complexes,

$$\pi^* : H^q\big(\Gamma(\Omega, \mathcal{C}^{n,*}(\pi_* L))\big) \longrightarrow H^q\big(\Gamma(\pi^{-1}(\Omega), \mathcal{C}^{n,*}_\sigma(L))\big), \tag{18}$$

for any open set $\Omega \subset X$ and all $q \geq 0$. If X is compact and we choose $\Omega = X$, then the left hand side in (18) is $H^{n,q}_{(2)}(X \setminus \operatorname{Sing} X, \pi_* L)$ for $\overline{\partial}_w(X \setminus \operatorname{Sing} X)$ is the $\overline{\partial}$-operator in the sense of distributions on $X \setminus \operatorname{Sing} X$, and the right hand side is just $H^{n,q}_{(2)}(M, L)$. This defines (7), (8) and (15), respectively.

We will now use Takegoshi's vanishing theorem [T] to show that the lower line in the commutative diagram is also exact if L is locally semi-positive with respect to the base space X. This will yield that (18) is in fact an isomorphism and that implies in particular Theorem 2.1.

Before, we shall mention another implication of the commutative diagram. The vertical arrow on the left hand side is an isomorphism because $\mathcal{L}^{n,0}(\pi_* L) \cong \pi_*(\mathcal{L}^{n,0}_\sigma(L))$ and the $\overline{\partial}_w$-equation extends over the exceptional set as described above (the L^2-extension [R1], Theorem 3.2). So,

$$\pi_*\big(\mathcal{K}_M(L)\big) \cong \mathcal{K}_X(\pi_* L). \tag{19}$$

Thus, $\mathcal{K}_X$ is in fact the canonical sheaf of Grauert–Riemenschneider as introduced in [GR]. It only remains to relate the direct image of the fine resolution (16) of $\mathcal{K}_M(L)$ to the fine resolution (10) of $\mathcal{K}_X(\pi_* L)$. This can be done by use of Takegoshi's vanishing theorem (see [T], Remark 2) which tells us that the higher direct image sheaves of $\mathcal{K}_M(L)$ do vanish:

$$R^q \pi_*\big(\mathcal{K}_M(L)\big) = 0, \quad q > 0. \tag{20}$$

Since (16) is exact, (20) implies that the lower line of the commutative diagram is another fine resolution of $\pi_*(\mathcal{K}_M(L)) \cong \mathcal{K}_X(\pi_* L)$ (use also (19)). But then

$$H^q(\Omega, \mathcal{K}_X(\pi_* L)) \cong H^q(\Omega, \pi_*(\mathcal{K}_M(L))) \cong H^q(\pi^{-1}(\Omega), \mathcal{K}_M(L))$$

and (18) is an isomorphism for all open sets $\Omega \subset X$ and all $q \geq 0$. If X is compact, the choice $\Omega = X$ proves Theorem 2.1.

3. The adjunction formula

3.1. **The adjunction formula for a smooth divisor.** Let M be a complex manifold of dimension n, and V a smooth hypersurface in M. By a slight abuse of notation, we call $[V]$ the normal bundle of V in M, where $[V]$ is the holomorphic line bundle such that the holomorphic sections in $[V]$ correspond to sections in $\mathcal{O}(V)$. The well-known adjunction formula states that

$$\iota_* \mathcal{K}_V \cong \mathcal{K}_M \otimes \mathcal{O}(V) / \mathcal{K}_M = \mathcal{K}_M([V]) / \mathcal{K}_M \tag{21}$$

in our notation from the last section, where $\iota : V \hookrightarrow M$ denotes the natural inclusion, i.e. $\iota_* \mathcal{K}_V$ is the trivial extension of the canonical sheaf $\mathcal{K}_V$ to M. If we denote the canonical bundles on V and M by K_V and K_M, respectively, then the adjunction formula can be expressed as

$$K_V \cong \big(K_M \otimes [V]\big)\big|_V .$$

We shall explain how the isomorphism in (21) can be realized explicitly. For further use, we take a slightly more general point of view. Let $\{f_j\}_j$ be a system of holomorphic functions defining V and $\omega \in C^\infty_{n,q}(M,[V])$ a smooth (n,q)-form with values in $[V]$; we identify ω with a semi-meromorphic (n,q)-form with at most a single pole along V. In local coordinates $z_1, ..., z_n$, we can write

$$\omega = \frac{g_j}{f_j} \wedge dz_1 \wedge \cdots \wedge dz_n,$$

where the g_j are smooth $(0,q)$-forms which transform as $g_j = (f_j/f_k) g_k$. We can now define the adjunction morphism locally as follows. For each point $p \in V$, there exists an f_j with $df_j \neq 0$ in a neighborhood of p, i.e. $\partial f_j/\partial z_\mu \neq 0$ in a neighborhood of the point p for some $1 \leq \mu \leq n$. In this neighborhood, we define the adjunction morphism as follows:

$$\omega \mapsto \omega' := (-1)^{q+\mu-1} g_j \wedge \left.\frac{dz_1 \wedge \cdots \wedge \widehat{dz_\mu} \wedge \cdots \wedge dz_n}{\partial f_j/\partial z_\mu}\right|_V . \tag{22}$$

This assignment does not depend on the choice of μ because the pull-back of $df_j = 0$ to V, i.e. $\iota^*(\sum_\mu \partial f_j/\partial z_\mu \, dz_\mu) = 0$. Moreover (using $f_j = (f_j/f_k) \cdot f_k$),

$$\frac{\partial f_j}{\partial z_\mu} = \frac{f_j}{f_k} \cdot \frac{\partial f_k}{\partial z_\mu} + f_k \cdot \frac{\partial}{\partial z_\mu}\left(\frac{f_j}{f_k}\right).$$

Hence, the $\partial f_j/\partial z_\mu$ transform as the g_j on V (where $f_k \equiv 0$). So, the assignment (22) does also not depend on the choice of the local defining function f_j. It is also easy to check that the mapping does not depend on the choice of local coordinates $z_1, ..., z_n$. Thus, (22) gives a well-defined mapping

$$\Psi : C^\infty_{n,q}(M,[V]) \to C^\infty_{n-1,q}(V) \ , \ \omega \mapsto \omega',$$

which we call the adjunction morphism. Note that Ψ maps holomorphic n-forms with values in $[V]$ on M to holomorphic $(n-1)$-forms on V. It is clear that $\Psi|_V$ is an isomorphism on V. This proves the adjunction formula (21). Note that on V, we have locally:

$$\omega = \frac{df_j}{f_j} \wedge \omega' = \frac{df_j}{f_j} \wedge \Psi(\omega), \tag{23}$$

which can be actually used to define Ψ. It follows directly from the definition that $\Psi \circ \overline{\partial} = \overline{\partial} \circ \Psi$ so that the adjunction morphism defines an adjunction map also on the level of cohomology classes

$$\Psi_V : H^{n,q}(M,[V]) \to H^{n-1,q}(V), \tag{24}$$

which we sometimes denote by Ψ_V to indicate the dependence on V.

Remark 3.1. The question whether $\overline{\partial}$-cohomology classes on V extend to M or not amounts to the question whether the map in (24) is surjective. This has a nice

cohomological realization. From the considerations above, we obtain the short exact sequence

$$0 \longrightarrow \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \longrightarrow \iota_* \mathcal{K}_V \longrightarrow 0.$$

By use of the long exact cohomology sequence, it follows that the induced map

$$H^q(M, \mathcal{K}_M \otimes \mathcal{O}(V)) \longrightarrow H^q(M, \iota_* \mathcal{K}_V) \cong H^q(V, \mathcal{K}_V)$$

is surjective if $H^{q+1}(M, \mathcal{K}_M) \cong H^{n,q+1}(M) = 0$.

3.2. **The adjunction formula for a singular divisor.** Let M be a complex manifold of dimension n, but V a hypersurface in M which is not necessarily smooth. Two problems occur. First, it is not clear what we mean by a canonical sheaf on V. Second, the adjunction morphism cannot be defined in a 'smooth' way as above because it will happen that $df_j = 0$ in singular points of V.

We can overcome these problems by using a bimeromorphically invariant form of the adjunction formula. Let

$$\pi : (V', M') \to (V, M)$$

be an embedded resolution of V in M, i.e. $\pi : M' \to M$ is a surjective proper holomorphic map such that $\pi|_{M' \setminus E} : M' \setminus E \to M \setminus \operatorname{Sing} V$ is a biholomorphism, where E is the exceptional divisor which consists of normal crossings only, and the regular hypersurface V' is the strict transform of V (see e.g. [BM], Theorem 13.2).

Hence, $\pi|_{V'} : V' \to V$ is a resolution of singularities for V. We denote $\pi|_{V'}$ again by π for ease of notation, that does not cause confusion.

The adjunction formula (21) for the pair (V', M') tells us that

$$0 \longrightarrow \mathcal{K}_{M'} \hookrightarrow \mathcal{K}_{M'} \otimes \mathcal{O}(V') \longrightarrow \iota_* \mathcal{K}_{V'} \longrightarrow 0$$

is exact. As we have already seen, (20), $R^q \pi_* \mathcal{K}_{M'} = 0$ for $q > 0$ (Takegoshi's vanishing theorem). So, we obtain the short exact sequence

$$0 \longrightarrow \pi_* \mathcal{K}_{M'} \hookrightarrow \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V') \big) \longrightarrow \pi_* \iota_* \mathcal{K}_{V'} \longrightarrow 0$$

on M. But $\pi_* \mathcal{K}_{M'} \cong \mathcal{K}_M$ (induced by pull-back of forms under π), and

$$\pi_* \iota_* \mathcal{K}_{V'} = \iota_* \pi_* \mathcal{K}_{V'} \cong \iota_* \mathcal{K}_V,$$

where $\iota_* \mathcal{K}_V$ is the trivial extension of $\mathcal{K}_V$ from V to M (letting $\iota : V \hookrightarrow M$ also the natural embedding).

We thus get the short exact sequence

$$0 \longrightarrow \mathcal{K}_M \hookrightarrow \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V') \big) \longrightarrow \iota_* \mathcal{K}_V \longrightarrow 0 \tag{25}$$

on M, and obtain the following **adjunction formula for the Grauert-Riemenschneider canonical sheaf** on a singular hypersurface:

$$\iota_* \mathcal{K}_V \cong \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V') \big) / \mathcal{K}_M \tag{26}$$

It is interesting to clarify the connection between $\mathcal{K}_M \otimes \mathcal{O}(V)$ and the direct image sheaf $\pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V') \big)$. We claim that there exists a **multiplier ideal sheaf** $\mathcal{J}_\pi(V)$ such that

$$\pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V') \big) \cong \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V). \tag{27}$$

Since $\pi_* \mathcal{K}_{M'} \cong \mathcal{K}_M$, it is clear that (27) holds with $\mathcal{J}_\pi(V)_x = \mathcal{O}_{M,x}$ for points $x \notin \operatorname{Sing} V$. So, consider a point $x \in \operatorname{Sing} V$. There exists a holomorphic function f in a neighborhood U_x of the point x defining the hypersurface V, i.e. $V = (f)$. Consider the pullback $\pi^* f = f \circ \pi$ on $\pi^{-1}(U_x)$. Then $\pi^* f$ is vanishing precisely of

order 1 on the strict transform V' of V because f is vanishing precisely of order 1 on the regular part of V. Let

$$(\pi^* f) = V' + E_f,$$

where E_f is a divisor with support on the exceptional set of the embedded resolution $\pi : M' \to M$. E_f is a normal crossing divisor since the exceptional set E has only normal crossings. $\mathcal{O}(-E_f)$ is the sheaf of germs of holomorphic functions which vanish at least to the order of $\pi^* f$ on E. For the direct image sheaf, we have that

$$\pi_* \mathcal{O}(-E_f) \subset \pi_* \mathcal{O}_{M'} \cong \mathcal{O}_M$$

on U_x, i.e. $\pi_* \mathcal{O}(-E_f)$ can be considered as a coherent sheaf of ideals in $\mathcal{O}_M$. So, there exist holomorphic functions $g_1, \ldots, g_k \in \mathcal{O}(U_x)$ that generate the direct image sheaf on a neighborhood of the point x. By restricting U_x, we can assume that this is the case on U_x. Then $\pi^* g_j \in \mathcal{O}(-E_f)(\pi^{-1}(U_x))$ for $j = 1, ..., k$, i.e. all the $\pi^* g_j$ vanish at least to the order of $\pi^* f$ on the exceptional set E, and the common zero set of the $\pi^* g_j$ is contained in E. On the other hand, f is in the direct image sheaf, and so $f = h_1 g_1 + ... + h_k g_k$ for holomorphic functions $h_1, ..., h_k \in \mathcal{O}(U_x)$. Hence,

$$\pi^* f = \pi^* h_1 \pi^* g_1 + ... + \pi^* h_k \pi^* g_k,$$

meaning that not all the $\pi^* g_j$ can vanish to an order strictly higher than $\pi^* f$ on any irreducible component of E. Thus, we conclude that

$$E_f = (\pi^* g_1, ..., \pi^* g_k), \tag{28}$$

i.e. the $\pi^* g_1$, ..., $\pi^* g_k$ generate $\mathcal{O}(-E_f)$. Let $\varphi := \log \big(|g_1| + ... + |g_k|\big)$ and let $\mathcal{J}(\varphi)$ be the multiplier ideal sheaf associated to the plurisubharmonic weight φ on U_x, i.e. the sheaf of germs of holomorphic functions F such that $|F|^2 e^{-2\varphi}$ is locally integrable. On the other hand, $\mathcal{J}(\pi^* \varphi)$ is the sheaf of germs of holomorphic functions H such that $H/(|\pi^* g_1| + ... + |\pi^* g_k|)$ is locally square-integrable. But E has only normal crossings, and so (28) implies that

$$\mathcal{J}(\pi^* \varphi) = \mathcal{O}(-E_f) \tag{29}$$

on $\pi^{-1}(U_x)$. We are now in the position to prove:

Theorem 3.2. *On U_x, the pull-back of forms under π induces the isomorphism*

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big).$$

Proof. It is well-known that

$$\pi^* : \mathcal{K}_M \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{J}(\pi^* \varphi)\big),$$

see e.g. [D3], Proposition 15.5. This is just the transformation law which behaves very well for $(n,0)$-forms. We will discuss that point in more detail in Section 4.2. By use of (29), we obtain

$$\pi^* : \mathcal{K}_M \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(-E_f)\big). \tag{30}$$

Let W be an open set in U_x. Then $\mathcal{K}_M \otimes \mathcal{J}(\varphi)(W)$ are the holomorphic n-forms on W with coefficients in $\mathcal{J}(\varphi)$, and $\pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(-E_f)\big)(W)$ are the holomorphic n-forms on $\pi^{-1}(W)$ with coefficients in $\mathcal{O}(-E_f)$. The isomorphism in (30) is given by pull-back of n-forms under π. So, the isomorphism (30) is preserved if we multiply the germs on the left-hand side by $1/f$, and the germs on the right-hand side by $1/\pi^* f$. Recalling that $(f) = V$ and $(\pi^* f) = V' + E_f$, it follows that

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_* \big(\mathcal{K}_{M'} \otimes \mathcal{O}(V' + E_f) \otimes \mathcal{O}(-E_f)\big),$$

meaning nothing else but

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(\varphi) \xrightarrow{\cong} \pi_*\big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big).$$

□

Note that (30) shows that $\mathcal{J}(\varphi)$ does not depend on the specific choice of φ. Since $\mathcal{K}_M$ and $\mathcal{O}(V)$ are invertible sheaves, we conclude the following result which together with (25) proves Theorem 1.2.

Theorem 3.3. *Let*

$$\mathcal{J}_\pi(V) := \mathcal{O}(V)^{-1} \otimes \mathcal{K}_M^{-1} \otimes \pi_*\big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big).$$

Then $\mathcal{J}_\pi(V)$ is a multiplier ideal sheaf such that pull-back of forms under π induces the isomorphism

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V) \xrightarrow{\cong} \pi_*\big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big).$$

Locally, $\mathcal{J}_\pi(V) = \mathcal{J}(\varphi)$ for a plurisubharmonic function

$$\varphi = \log\big(|g_1| + ... + |g_k|\big),$$

where the $g_1, ..., g_k$ are holomorphic functions such that

$$(\pi^* f)|_E = E_f = (\pi^* g_1, ..., \pi^* g_k).$$

By (25) and (26), we obtain the short exact sequence

$$0 \longrightarrow \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V) \longrightarrow \iota_* \mathcal{K}_V \longrightarrow 0 \tag{31}$$

and the **adjunction formula for the Grauert-Riemenschneider canonical sheaf on a singular hypersurface**:

$$\iota_* \mathcal{K}_V \cong \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}_\pi(V) \big/ \mathcal{K}_M. \tag{32}$$

Since $\mathcal{K}_M$ and $\mathcal{O}(V)$ are invertible, it follows from (31) that $\mathcal{J}_\pi(V)$ does not depend on the resolution π. So, we write $\mathcal{J}(V) = \mathcal{J}_\pi(V)$ and call $\mathcal{O}(V) \otimes \mathcal{J}(V)$ the **adjunction sheaf of V in M**.

Note that the natural injection in (31) makes sense since $\mathcal{O}_M \subset \mathcal{O}(V) \otimes \mathcal{J}(V)$, that $\mathcal{J}(V)$ is coherent, and that the zero set of $\mathcal{J}(V)$ is contained in $\operatorname{Sing} V$.

Remark 3.4. As in Remark 3.1 we obtain from (31) by use of the long exact cohomology sequence the adjunction map for the flabby cohomology

$$H^q\big(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big) \longrightarrow H^q(M, \iota_* \mathcal{K}_V) \cong H^q(V, \mathcal{K}_V), \tag{33}$$

which is surjective e.g. if $H^{q+1}(M, \mathcal{K}_M) \cong H^{n,q+1}(M) = 0$.

3.3. **Relation to the Grothendieck dualizing sheaf.** As mentioned in the introduction, one can also consider the adjunction formula for Grothendieck's dualizing sheaf. As above, let M be a complex manifold of dimension n and V a hypersurface in M. We denote by

$$\omega_V := \mathcal{E}xt^1_{\mathcal{O}_M}(\mathcal{O}_V, \mathcal{K}_M)$$

Grothendieck's dualizing sheaf (see e.g. [PR], §5.3). As V is a complete intersection, it follows (see also [PR], §5.3) that

$$\omega_V \cong \big(\mathcal{K}_M \otimes \mathcal{O}(V)\big)\big|_V.$$

In other words, we obtain the natural short exact sequence

$$0 \to \mathcal{K}_M \hookrightarrow \mathcal{K}_M \otimes \mathcal{O}(V) \to \iota_* \omega_V \to 0,$$

where $\iota : V \hookrightarrow M$ denotes again the natural embedding.

Combining that with Theorem 3.3, we get the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K}_M \arrow[r] \arrow[d, "="] & \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \arrow[r] \arrow[d, "j"] & \iota_* \mathcal{K}_V \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \mathcal{K}_M \arrow[r] & \mathcal{K}_M \otimes \mathcal{O}(V) \arrow[r] & \iota_* \omega_V \arrow[r] & 0
\end{tikzcd} \tag{34}$$

where the map j is the natural inclusion. The diagram (34) induces a natural isomorphism

$$\iota_* \mathcal{K}_V \cong \big(\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big)/\mathcal{K}_M \cong \mathcal{J}(V) \otimes \big(\mathcal{K}_M \otimes \mathcal{O}(V)\big)/\mathcal{K}_M \cong \mathcal{J}(V) \otimes \omega_V.$$

As $\mathcal{J}(V)|_V \subset \mathcal{O}_V$, this implies particularly that there is a natural inclusion of the Grauert-Riemenschneider canonical sheaf into Grothendieck's dualizing sheaf, $\mathcal{K}_V \subset \omega_V$. This proves Theorem 1.3. For a normal V, the inclusion $\mathcal{K}_V \subset \omega_V$ was proved in [GR].

Let us now prove Theorem 1.4; assume therefore that V is a normal. We first need to recall what is meant by canonical singularities. As V is a normal hypersurface, $\omega_V \cong \big(\mathcal{K}_M \otimes \mathcal{O}(V)\big)|_V$ is an invertible sheaf corresponding to a canonical Cartier divisor K_V, in particular V is Gorenstein (see e.g. [PR], §5.4). Then V has canonical singularities if the following condition holds (see [K1], Section 3): If $\pi : N \to V$ is any resolution of singularities and K_N is the canonical divisor of N, so that we can write

$$K_N = \pi^* K_V + \sum a_j E_j, \tag{35}$$

where the E_j are the irreducible components of the exceptional divisor and the a_j are rational coefficients. Then $a_j \geq 0$ for all indices j.

Thus, V has canonical singularities precisely if $\pi^*\omega_V \subset \mathcal{K}_N$ for any resolution of singularities $\pi : N \to V$ (as (35) is equivalent to $\mathcal{K}_N = \pi^*\omega_V \otimes \mathcal{O}(\sum a_j E_j)$ where $\sum a_j E_j$ is an effective divisor).

It is now not hard to prove Theorem 1.4. Recall that $\mathcal{K}_V = \omega_V$, cf. (19). Let $\pi : N \to V$ be any resolution of singularities. Then $\mathcal{K}_V = \pi_* \mathcal{K}_N$ by definition and we deduce:

$$\pi^*\omega_V = \pi^* \mathcal{K}_V = \pi^* \pi_* \mathcal{K}_N \subset \mathcal{K}_N,$$

and so V has canonical singularities.

Conversely, assume that V has canonical singularities. By Theorem 1.3, we just have to show that $\omega_V \subset \mathcal{K}_V$. Let $\pi : N \to V$ be any resolution of singularities. Then $\mathcal{K}_V = \pi_* \mathcal{K}_N$. As V has canonical singularities, we know that $\pi^*\omega_V \subset \mathcal{K}_N$. But then:

$$\omega_V \subset \pi_* \pi^* \omega_V \subset \pi_* \mathcal{K}_N = \mathcal{K}_V.$$

3.4. **The commutative adjunction diagram.** It is our next purpose to give an explicit realization of the adjunction map (33) on the level of (n,q)-forms. We can achieve that easily by tensoring (31) with the sheaf of germs of smooth $(0,q)$-forms $\mathcal{C}^\infty_{0,q}$.

Let (M,h) be a compact Hermitian manifold, V a singular hypersurface in M, and $\pi : (V', M') \to (V, M)$ an embedded resolution of V in M as above. As in Section 2.3, let $\gamma := \pi^* h$ be the pullback of the Hermitian metric h of M to M'. Then γ is a positive semidefinite pseudo-metric with degeneracy locus E. Again, give M' any positive definite metric σ such that $\gamma \lesssim \sigma$. As in Section 3.1, let $\{f'_j\}_j$ be a finite defining system for V' in M'.

We are now in the position to describe an adjunction morphism on M similar to the procedure in Section 3.1. On M, we have to replace the normal bundle $[V]$ by the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

Consider the adjunction map

$$\Psi' : \mathcal{K}_{M'} \otimes \mathcal{O}(V') \longrightarrow \iota_* \mathcal{K}_{V'}$$

for the non-singular hypersurface V' in M' defined as in Section 3.1. Recall that we denote both the natural inclusions $V' \hookrightarrow M'$, $V \hookrightarrow M$ by ι, and $\pi|_{V'}$ by π for ease of notation. Since

$$\pi^* : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \xrightarrow{\cong} \pi_*\big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big)$$

and $\pi^* : \mathcal{K}_V \xrightarrow{\cong} \pi_* \mathcal{K}_{V'}$, we obtain the first commutative adjunction diagram

$$\begin{array}{ccc} \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) & \xrightarrow{\Psi} & \iota_* \mathcal{K}_V \\ \cong \big\downarrow \pi^* & & \cong \big\downarrow \pi^* \\ \pi_*\big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big) & \xrightarrow{\Psi'} & \pi_* \iota_* \mathcal{K}_{V'} \end{array} \tag{36}$$

defining the adjunction map Ψ on M. Note that $\pi_* \iota_* \mathcal{K}_{V'} = \iota_* \pi_* \mathcal{K}_{V'}$ and that the kernel of Ψ is just the natural inclusion of $\mathcal{K}_M$ in $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$. So, Ψ realizes the isomorphism (32), i.e.

$$\Psi : \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \big/ \mathcal{K}_M \xrightarrow{\cong} \iota_* \mathcal{K}_V.$$

Considering the maps on global sections, (36) yields the commutative diagram

$$\begin{array}{ccc} \Gamma\big(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big) & \xrightarrow{\Psi} & \Gamma(V, \mathcal{K}_V) \\ \cong \big\downarrow \pi^* & & \cong \big\downarrow \pi^* \\ \Gamma\big(M', \mathcal{K}_{M'} \otimes \mathcal{O}(V')\big) & \xrightarrow{\Psi'} & \Gamma(V', \mathcal{K}_{V'}) \end{array}$$

We can now describe Ψ explicitly. Let $\omega \in \Gamma(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$. By use of (23), we get that

$$\pi^* \omega = \frac{df'_j}{f'_j} \wedge \Psi'(\pi^* \omega) = \frac{df'_j}{f'_j} \wedge \pi^* \Psi(\omega) \tag{37}$$

locally on V'.

Now then, let f be a local defining function for V in M. Then $\pi^* f$ is vanishing precisely to order 1 on the resolution V' of V. Hence

$$\pi^* \left(\frac{df}{f} \right) = \frac{\pi^* df}{\pi^* f} = \frac{df'_j}{f'_j}$$

as $(1,0)$-forms in $\mathcal{O}(V')|_{V' \setminus E}$. Since π is a biholomorphism on $M' \setminus E$, it follows from (37) that

$$\omega = \frac{df}{f} \wedge \Psi(\omega) \tag{38}$$

on $\operatorname{Reg} V$, where f is a local defining function for V in M. This shows that the adjunction map Ψ does not depend on the resolution $\pi : M' \to M$ since df does not vanish on $\operatorname{Reg} V$.

Let us now define the adjunction map for (germs of) (n,q)-forms. That can be done simply by tensoring the diagram (36) by the sheaves of germs of smooth $(0,q)$-forms, so that we obtain the commutative diagram

$$\begin{tikzcd} \mathcal{C}^\infty_{0,q} \otimes \big(\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big) \arrow[r, "{(1,\Psi)}"] \arrow[d, "{(\pi^*,\pi^*)}"] & \mathcal{C}^\infty_{0,q} \otimes \iota_*\mathcal{K}_V \arrow[d, "{(\pi^*,\pi^*)}"] \\ \pi_*\mathcal{C}^\infty_{0,q} \otimes \pi_*\big(\mathcal{K}_{M'} \otimes \mathcal{O}(V')\big) \arrow[r, "{(1,\Psi')}"] & \pi_*\mathcal{C}^\infty_{0,q} \otimes \pi_*\iota_*\mathcal{K}_{V'} \end{tikzcd} \tag{39}$$

where the vertical arrows are not isomorphisms any more.

It is easy to see that we can complement the diagram on the right hand side by natural mappings to the sheaves of germs of L^2-forms in the domain of $\overline{\partial}$ on V and V', respectively:

$$\begin{tikzcd} \mathcal{C}^\infty_{0,q} \otimes \iota_*\mathcal{K}_V \arrow[r] \arrow[d, "{(\pi^*,\pi^*)}"] & \iota_*\mathcal{C}^{n-1,q}_V \arrow[d, "\pi^*"] \\ \pi_*\mathcal{C}^\infty_{0,q} \otimes \pi_*\iota_*\mathcal{K}_{V'} \arrow[r] & \pi_*\iota_*\mathcal{C}^{n-1,q}_{V'} \end{tikzcd} \tag{40}$$

Merging (39), (40) and adopting the notation, we obtain the commutative adjunction diagram for (germs of) (n,q)-forms:

$$\begin{tikzcd} \mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \arrow[r, "\Psi"] \arrow[d, "\pi^*"] & \iota_*\mathcal{C}^{n-1,q}_V \arrow[d, "\pi^*"] \\ \pi_*\big(\mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V')\big) \arrow[r, "\Psi'"] & \pi_*\iota_*\mathcal{C}^{n-1,q}_{V'} \end{tikzcd}$$

For global sections, we have the commutative diagram

$$\begin{tikzcd} \Gamma\big(M, \mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big) \arrow[r, "\Psi"] \arrow[d, "\pi^*"] & \Gamma(V, \mathcal{C}^{n-1,q}_V) \arrow[d, "\pi^*"] \\ \Gamma\big(M', \mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V')\big) \arrow[r, "\Psi'"] & \Gamma(V', \mathcal{C}^{n-1,q}_{V'}) \end{tikzcd} \tag{41}$$

Let $\omega \in \Gamma(M, \mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$. It follows from (38) that we still have

$$\omega = \frac{df}{f} \wedge \Psi(\omega) \tag{42}$$

on $\operatorname{Reg} V$ when f is a local defining function for V in M.

Since $\overline{\partial}$ commutes with π^* and Ψ', it must also commute with Ψ, and so we deduce from (41) the commutative diagram on the level of $\overline{\partial}$-cohomology:

$$\begin{tikzcd} H^q\big(\Gamma(M, \mathcal{C}^\infty_{n,*} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))\big) \arrow[r, "\Psi_V"] \arrow[d, "\pi^*"] & H^q\big(\Gamma(V, \mathcal{C}^{n-1,*}_V)\big) \arrow[d, "\pi^*"] \\ H^q\big(\Gamma(M', \mathcal{C}^\infty_{n,*} \otimes \mathcal{O}(V'))\big) \arrow[r, "\Psi_{V'}"] & H^q\big(\Gamma(V', \mathcal{C}^{n-1,*}_{V'})\big) \end{tikzcd} \tag{43}$$

Note that the vertical map on the right-hand side was already defined in (18) and that the groups on the right hand-side are by definition the L^2-cohomology groups for the $\overline{\partial}_w$-operator.

It does not cause any additional difficulty to define the adjunction morphisms as above also for (n,q)-forms with values in a Hermitian line bundle $F \to M$ and in $\pi^* F \to M'$, respectively. We get:

Theorem 3.5. *Let M be a compact Hermitian manifold of dimension n, $V \subset M$ a singular hypersurface in M, and $F \to M$ a Hermitian line bundle. Let $\pi : (V', M') \to (V, M)$ be an embedded resolution of singularities of V in M as above, and let $\mathcal{J}(V)$ be the multiplier ideal sheaf as defined in Theorem 3.3, i.e. $\mathcal{O}(V) \otimes \mathcal{J}(V)$ is the adjunction sheaf for the divisor V in M. Then there exists a commutative diagram*

$$\begin{array}{ccc} H^q\big(\Gamma(M, \mathcal{C}^\infty_{n,*}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))\big) & \xrightarrow{\Psi_V} & H^{n-1,q}_{(2)}(V^*, F) \\ \Big\downarrow \pi^* & & \cong \Big\downarrow \pi^* \\ H^q\big(\Gamma(M', \mathcal{C}^\infty_{n,*}(\pi^* F) \otimes \mathcal{O}(V')\big) & \xrightarrow{\Psi_{V'}} & H^{n-1,q}_{(2)}(V', \pi^* F), \end{array} \tag{44}$$

where $\Psi_{V'}$ is the usual adjunction map for the smooth divisor V' in M', and Ψ_V is the adjunction map for the non-smooth divisor V in M as defined in (43).

The vertical maps π^* in (44) are induced by pull-back of forms under π, the vertical map on the right-hand side is an isomorphism by Theorem 2.1, and the vertical map on the left-hand side is surjective by Corollary 4.7 (see below).

Note that the lower line in (44) can be understood as

$$\Psi_{V'} : H^{n,q}(M', \pi^* F \otimes [V']) \longrightarrow H^{n-1,q}_{(2)}(V', \pi^* F),$$

where $[V'] \to M'$ is the normal bundle of V' in M' (see Section 3.1).

Our purpose is to determine conditions under which the adjunction map Ψ_V in the upper line of the commutative diagram (44) is surjective. For this, it would be interesting to know whether the vertical map on the left-hand side of the diagram (44) is also an isomorphism. We will see later (Theorem 4.6) that this is actually true if we replace the upper left corner of the diagram by some L^2-cohomology group. Before studying L^2-cohomology in the next section, let us first investigate the case of C^∞-cohomology in (44) a bit closer.

It is well known (just solve the $\overline{\partial}$-equation locally in the C^∞-category for forms with values in a line bundle) that the complex

$$\big(\mathcal{C}^\infty_{n,*}(\pi^* F \otimes [V']), \overline{\partial}\big)$$

is a fine resolution of $\mathcal{K}_{M'}(\pi^* F \otimes [V'])$, hence

$$H^q\big(\Gamma(M', \mathcal{C}^\infty_{n,*}(\pi^* F \otimes [V'])\big) \cong H^q\big(M', \mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big). \tag{45}$$

On the other hand, we have already seen that pull-back of forms under π induces the isomorphism

$$\pi^* : \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \xrightarrow{\cong} \pi_*\big(\mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big),$$

since the line-bundle $F \to M$ is added easily to the statement of Theorem 3.2.

Lemma 3.6. *If $U \subset M$ is sufficiently small, then $\big(\pi^* F \otimes [V']\big)|_{\pi^{-1}(U)}$ is semi-positive.*

Proof. The factor $\pi^* F$ is irrelevant since the statement is local with respect to M and semi-positivity is stable under pullback by holomorphic mappings. Let $f \in \mathcal{O}(U)$ be a defining function for $V|_U$ so that

$$(\pi^* f) = V'|_{\pi^{-1}(U)} + E_f|_{\pi^{-1}(U)},$$

where E_f is a divisor with support on the exceptional set of the resolution. Then

$$[V']|_{\pi^{-1}(U)} = \pi^*[V] \otimes [-E_f]|_{\pi^{-1}(U)}$$

and so, by the argument above, it is sufficient to see that $[-E_f]|_{\pi^{-1}(U)}$ is positive.

Take a covering $\{U_\alpha\}$ of $\pi^{-1}(U)$ such that $\pi^* f = f^0_\alpha \cdot f'_\alpha$ in U_α, where $(f^0_\alpha) = V'|_{U_\alpha}$ and $(f'_\alpha) = E_f|_{U_\alpha}$. Recall from Section 3.2 that (possibly after shrinking U) there are $g_1, \dots, g_k \in \mathcal{O}(U)$ that generate the direct image of $\mathcal{O}(-E_f)$. It follows that there is a non-vanishing tuple $h_\alpha = (h_{1\alpha}, \dots, h_{k\alpha}) \in \mathcal{O}(U_\alpha)^k$ such that

$$\pi^* g = (\pi^* g_1, \dots, \pi^* g_k) = f'_\alpha h_\alpha \quad \text{in} \quad U_\alpha.$$

Letting $|h_\alpha| = (|h_{1\alpha}|^2 + \cdots + |h_{k\alpha}|^2)^{1/2}$ it is straight forward to check that the local functions $e^{-\log|h_\alpha|}$ transforms as a metric on $[-E_f]$. Since $\log|h_\alpha|$ is plurisubharmonic $[-E_f]|_{\pi^{-1}(U)}$ is positive. □

Thus, $\pi^* F \otimes [V']$ is locally semi-positive with respect to M. Since $\mathcal{K}_{M'}(\pi^* F \otimes [V']) \cong \mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')$ we conclude by Takegoshi's vanishing theorem (see [T], Remark 2) as in the proof of Theorem 2.1 (see (20)):

Theorem 3.7.

$$R^q \pi_* \big(\mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big) = 0 \qquad \textit{for} \quad q > 0.$$

So, the direct image complex

$$\big(\pi_* \big(\mathcal{C}^\infty_{n,*}(\pi^* F) \otimes \mathcal{O}(V')\big), \pi_* \overline{\partial}\big)$$

is a fine resolution of

$$\pi_* \big(\mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big) \cong \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V),$$

and

$$H^q(M', \mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')) \cong H^q(M, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)).$$

Combining that with (45), we would get that the vertical map on the left-hand side of (44) is an isomorphism if we knew that the complex

$$\big(\mathcal{C}^\infty_{n,*}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V), \overline{\partial}\big)$$

were exact. But this is a delicate problem for it involves the multiplier ideal sheaf $\mathcal{J}(V)$. We can prove the required local exactness only in the L^2-category. That will be done in the next section where we work with a nice L^2-resolution for $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$. We have to face the problem that the restriction of such L^2-forms to the hypersurface V need not be L^2 on the hypersurface. We will avoid that problem by choosing suitable smooth representatives of cohomology classes.

4. Extension of cohomology classes

4.1. **Metrics on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$.** As it is invertible outside a proper analytic subset of M, we may talk about a metric on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ in a certain sense; it will be a singular metric on the normal bundle $[V]$.

Let $F \to M$ be a Hermitian line bundle with metric $H = e^{-2\varphi}$. As usually, we use the notation $e^{-2\varphi}$ as if the weight φ were defined globally. Recall from Section 2.2 the definition of the sheaves $\mathcal{C}^{p,q}_M(F)$, i.e. $\mathcal{C}^{p,q}_M(F)$ is the sheaf of germs of measurable

(p,q)-forms u with values in F such that $|u|^2_{F,H} = |u|^2 e^{-2\varphi}$ and $|\overline{\partial}_w u|^2_{F,H} = |\overline{\partial}_w u|^2 e^{-2\varphi}$ both are locally integrable.

We do now allow explicitly that $H = e^{-2\varphi}$ is a singular Hermitian metric, i.e. the weight φ can be an arbitrary function in $L^{1,loc}$ (see [D3], Definition 11.20). When this case appears, we write F^{sing} to indicate that F carries a possibly singular metric. Recall that a multiplier ideal sheaf $\mathcal{J}(\psi)$ locally can be interpreted as the sheaf of holomorphic L^2-sections of the trivial line bundle with the singular metric $e^{-2\psi}$.

Recall from Definition 1.5 that a singular metric $e^{-2\psi}$ on the normal bundle $[V]$ of the hypersurface V in M is said to be smooth on the adjunction sheaf $\mathcal{O}(V)\otimes\mathcal{J}(V)$ if there exists an embedded resolution of singularities with only normal crossings $\pi : (V', M') \to (V, M)$ such that $e^{-2\pi^*\psi+2\varphi_E}$ is a smooth metric on the normal bundle $[V']$ of V' in M'.

We should first make sure that smooth metrics on adjunction sheaves do actually exist, but that follows easily from the definition of the adjunction sheaf:

Theorem 4.1. *Let*

$$\pi : (V', M') \to (V, M)$$

be any embedded resolution of singularities of V in M with only normal crossings, and $e^{-2\psi'}$ a smooth Hermitian metric on the normal bundle $[V']$ of V' in M'. Then $e^{-2\psi'}$ induces a singular metric $e^{-2\psi}$ on $[V]$ that is smooth on the adjunction sheaf $\mathcal{O}(V)\otimes\mathcal{J}(V)$. Moreover, if $[V']|_{\pi^{-1}(U)}$ is semi-positive, then $[V]|_U$ is too.

Proof. It is sufficient to prove the theorem locally in M so let $U \subset M$ such that $V|_U$ is defined by $f \in \mathcal{O}(U)$. Then $(\pi^* f) = V'|_{\pi^{-1}(U)} + E_f|_{\pi^{-1}(U)}$, where V' is the strict transform of V, and E_f is an effective divisor with support on the exceptional set E; cf. the proof of Lemma 3.6. Take a covering $\{U_\alpha\}$ of $\pi^{-1}(U)$ such that $\pi^* f = f^0_\alpha f'_\alpha$ in U_α where f^0_α defines $V'|_{U_\alpha}$ and f'_α defines $E_f|_{U_\alpha}$. Now, if s is any section of $[V]$ over U then $\pi^* s/f'_\alpha$ transforms as the f^0_α and hence defines a semi-meromorphic section of $[V']$ over U_α. We define the singular metric on $[V]$ by letting

$$\|s\|^2_V := |\pi^* s|^2/|f'_\alpha|^2 e^{-2\psi'_\alpha} = |\pi^* s|^2 e^{-2(\psi'_\alpha+\varphi_E)}, \tag{46}$$

where $e^{-2\psi'_\alpha}$ are the local functions for the metric on $[V']$. Clearly, this singular metric on $[V]$ induces the original metric $e^{-2\psi'}$ on $[V']$ and hence, the singular metric on $[V]$ is smooth on the adjunction sheaf $\mathcal{O}(V)\otimes\mathcal{J}(V)$ in the sense of Definition 1.5.

Assume now that $[V']|_{\pi^{-1}(U)}$ is semi-positive, i.e., that $dd^c\psi'_\alpha \geq 0$; notice that by Lemma 3.6 this can always be achieved if U is small enough. By (46), the singular metric $e^{-2\psi}$ on $[V]$ has the property that $\pi^*\psi = \psi'_\alpha + \log|f'_\alpha|$ and so, in the sense of currents,

$$dd^c\psi = \pi_*(dd^c\psi'_\alpha + T) = \pi_* dd^c\psi'_\alpha,$$

where T is the current of integration on $|E_f|$; the last equality follows since $\pi_* T$ is a normal $(1,1)$-current with support on $\operatorname{Sing} V$, which has codimension ≥ 2. Hence, $[V]|_U$ is semi-positive.

□

Let $e^{-2\psi}$ be the induced metric on $[V]$ from Theorem 4.1. With the notation from the proof above, we then have that $\pi^* e^{-2\psi} \sim |f'_\alpha|^{-2}$. Recall from Section 3.2 that we can choose holomorphic functions $g_1, ..., g_k \in \mathcal{O}(U)$ such that $g_1, ..., g_k$ generate the direct image of $\mathcal{O}(-E_f)$ over U, where U is a small open set in M, i.e. E_f is precisely the common zero set of $\pi^* g_1$, ..., $\pi^* g_k$ (counted with multiplicities). Hence,

$|f'_\alpha| \sim |\pi^* g_1| + \cdots + |\pi^* g_k|$, and so

$$e^{-2\psi} \sim \big(|g_1| + ... + |g_k|\big)^{-2} = e^{-2\varphi}, \tag{47}$$

where φ is a local defining function for the multiplier ideal sheaf $\mathcal{J}(V)$.

Theorem 4.2. *Let $[V]^{sing}$ be the normal bundle of a hypersurface V in M carrying a singular Hermitian metric which is a smooth metric on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ in the sense of Definition 1.5. Then*

$$\mathcal{O}(V) \otimes \mathcal{J}(V) \cong \ker \overline{\partial}_w \subset \mathcal{C}^{0,0}_M([V]^{sing}),$$

i.e. sections of the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ can be identified with square-integrable holomorphic sections of $[V]^{sing}$.

Proof. Let $e^{-2\psi}$ be the singular Hermitian metric on $[V]^{sing}$, which is a smooth metric on $\mathcal{O}(V) \otimes \mathcal{J}(V)$. By Definition 1.5, there exists an embedded resolution $\pi : (V', M') \to (V, M)$ of V in M with only normal crossings such that $\pi^* e^{-2\psi}$ induces a smooth metric on the normal bundle $[V']$ of V' in M'. But then, by (47), $\psi \sim \varphi$ where φ is a local defining function for the multiplier ideal sheaf $\mathcal{J}(V)$. Hence, a holomorphic section h of $[V]$ is in $\mathcal{C}^{0,0}_M([V]^{sing})$ precisely if $|h|^2 e^{-2\varphi}$ is locally integrable (in a trivialization of $[V]$). □

Remark 4.3. Let $e^{-2\psi}$ and $e^{-2\psi}$ be smooth metrics on $\mathcal{O}(V) \otimes \mathcal{J}(V)$. Then note, in light of (47), that being L^2 with respect to $e^{-2\psi}$ is equivalent to being L^2 with respect to $e^{-2\psi'}$.

4.2. **L^2-resolution for the adjunction sheaf.** By use of a smooth metric on $\mathcal{O}(V) \otimes \mathcal{J}(V)$, which always exists by Theorem 4.1, we can now introduce an L^2-resolution for $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ which is suitable for our purposes. As $\mathcal{K}_M$ is invertible, this is equivalent to an L^2-resolution for the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$, but in view of our later applications, we adhere to the statement in terms of (n, q)-forms.

Theorem 4.4. *Let $[V]^{sing}$ be the normal bundle of a hypersurface V in M carrying a singular Hermitian metric which is a smooth metric on the adjunction sheaf $\mathcal{O}(V) \otimes \mathcal{J}(V)$ in the sense of Definition 1.5. Then the complex $\big(\mathcal{C}^{n,*}_M([V]^{sing}), \overline{\partial}_w\big)$ is a fine resolution for $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$.*

The statement is essentially well known as it is part of the Nadel vanishing theorem (see [D3], Theorem 15.8).

Proof. Our Theorem 4.2 above implies that sections of $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ can be identified with square-integrable holomorphic sections of $[V]^{sing}$, i.e.

$$\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \cong \ker \overline{\partial}_w \subset \mathcal{C}^{n,0}_M([V]^{sing}).$$

By Theorem 4.1 and Remark 4.3 we may assume that $e^{-2\psi}$ is locally semi-positive, and positive on V_{reg}. It follows that exactness of the complex $\big(\mathcal{C}^{n,*}_M([V]^{sing}), \overline{\partial}_w\big)$ is equivalent to local L^2-exactness of the $\overline{\partial}$-equation for (n, q)-forms with values in a holomorphic line bundle with a singular Hermitian metric which is positive semi-definite (and positive on V_{reg}). But this is well-known (see [D3], Corollary 14.3, and the proof of the Nadel vanishing theorem, [D3], Theorem 15.8). It is furthermore clear that the sheaves $\mathcal{C}^{n,q}_M([V]^{sing})$ admit a smooth partition of unity, so that $\big(\mathcal{C}^{n,*}_M([V]^{sing}), \overline{\partial}_w\big)$ is in fact a fine resolution of $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$. □

It follows that the (flabby) cohomology of $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ can be expressed in terms of the L^2-cohomology of forms with values in the line bundle $[V]^{sing}$:

$$H^q(M, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \cong H^{n,q}_{(2)}(M, [V]^{sing}) := H^q\big(\Gamma(M, \mathcal{C}^{n,*}([V]^{sing}))\big). \quad (48)$$

4.3. **The adjunction diagram for L^2-cohomology classes.** Let $[V]^{sing}$ be the normal bundle of a hypersurface V with a singular Hermitian metric, smooth on $\mathcal{O}(V) \otimes \mathcal{J}(V)$. In order to define the adjunction map for L^2-cohomology classes with values in $[V]^{sing}$ we need the following representation. This is necessary as L^2-forms do not behave well under restriction to lower-dimensional subspaces.

Lemma 4.5. *Each L^2-cohomology class $[\phi] \in H^{n,q}_{(2)}(M, [V]^{sing})$ has a smooth representative $\phi \in \Gamma\big(M, \mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big)$.*

Proof. The statement follows by the DeRham-Weil-Dolbeault isomorphism (see Demailly, [D4], IV.6 The DeRham-Weil Isomorphism Theorem, or adopt the procedure from [H3], Chapter 7.4). Let $\mathcal{U} = \{U_\alpha\}$ be a Stein cover for M and $\{\chi_\alpha\}$ a smooth partition of unity subordinate to $\mathcal{U}$. Recall that the DeRham-Weil-Dolbeault map on Čech cohomology

$$[\Lambda_q] : \check{H}^q\big(\mathcal{U}, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big) \longrightarrow H^{n,q}_{(2)}(M, [V]^{sing}) \quad (49)$$

is defined as follows: given a Čech cocycle $c \in C^q(\mathcal{U}, \mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$, set

$$\Lambda_q c := \sum_{\nu_0,\dots,\nu_q} c_{\nu_0\cdots\nu_q} \chi_{\nu_q} \overline{\partial}\chi_{\nu_0} \wedge \cdots \wedge \overline{\partial}\chi_{\nu_{q-1}}. \quad (50)$$

As $(\mathcal{C}^{n,*}_M([V]^{sing}), \overline{\partial}_w)$ is a fine resolution for $\mathcal{K}_M \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ (see Theorem 4.4 and (48)), $[\Lambda_q]$ is an isomorphism. So, each class $[\phi] \in H^{n,q}_{(2)}(M, [V]^{sing})$ has a representative

$$\phi = \Lambda_q c \in \Gamma\big(M, \mathcal{C}^\infty_{n,q} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big).$$

□

We can now set up the commutative adjunction diagram for L^2-cohomology classes, replacing the upper left cohomology group in (44) by $H^{n,q}_{(2)}(M, [V]^{sing})$. Note that it does not cause any difficulty to include a Hermitian holomorphic line bundle $F \to M$ in the statement of Theorem 4.4 and Lemma 4.5.

Theorem 4.6. *Let M be a compact Hermitian manifold of dimension n, $V \subset M$ a singular hypersurface in M, $F \to M$ a Hermitian holomorphic line bundle, and let $\mathcal{J}(V)$ be the multiplier ideal sheaf as defined in Theorem 3.3, i.e. $\mathcal{O}(V) \otimes \mathcal{J}(V)$ is the adjunction sheaf for the non-smooth divisor V in M.*

Let $e^{-2\psi}$ be a singular Hermitian metric on the normal bundle $[V]$ of V in M which is a smooth metric on $\mathcal{O}(V) \otimes \mathcal{J}(V)$ according to Definition 1.5 so that the complex $(\mathcal{C}^{n,*}_M(F \otimes [V]^{sing}), \overline{\partial}_w)$ is a fine resolution for $\mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$.

Let

$$\pi : (V', M') \to (V, M)$$

be an embedded resolution of singularities of V in M such that union of the exceptional set and the strict transform V' of V has only normal crossings and such that $\pi^* e^{-2\psi}$ induces a smooth metric on the normal bundle $[V']$ of V' in M'.

Then there exists a commutative diagram

$$\begin{array}{ccc} H^{n,q}_{(2)}(M, F \otimes [V]^{sing}) & \xrightarrow{\Psi_V} & H^{n-1,q}_{(2)}(V^*, F) \\ \cong \Big\downarrow \pi^* & & \cong \Big\downarrow \pi^* \\ H^{n,q}(M', \pi^* F \otimes [V']) & \xrightarrow{\Psi_{V'}} & H^{n-1,q}_{(2)}(V', \pi^* F), \end{array} \tag{51}$$

where $\Psi_{V'}$ is the adjunction map for the smooth divisor V' in M', Ψ_V is induced by the adjunction map for the non-smooth divisor V in M as defined in (41).

The vertical maps π^* are induced by pull-back of forms under π and both are isomorphisms.

Proof. Starting from Theorem 3.5, we replace the cohomology group

$$H^q\big(\Gamma(M, \mathcal{C}^\infty_{n,*}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))\big)$$

in the upper left corner of the commutative diagram (44) by the L^2-cohomology

$$H^{n,q}_{(2)}(M, F \otimes [V]^{sing}) = H^q\big(\Gamma(M, \mathcal{C}^{n,*}_M(F \otimes [V]^{sing}))\big).$$

We do that by adding the map

$$\Lambda_q \circ [\Lambda_q]^{-1} : H^{n,q}_{(2)}(M, F \otimes [V]^{sing}) \longrightarrow H^q\big(\Gamma(M, \mathcal{C}^\infty_{n,*}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))\big)$$

to the diagram (44), where Λ_q is the DeRham-Weil-Dolbeault map as defined in (49), (50). The application of $\Lambda_q \circ [\Lambda_q]^{-1}$ means to choose smooth representatives in $\Gamma(M, \mathcal{C}^\infty_{n,q}(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V))$ for cohomology classes in $H^{n,q}_{(2)}(M, F \otimes [V]^{sing})$. Note that $\Lambda_q \circ [\Lambda_q]^{-1}$ does not depend on the choices made in Lemma 4.5 as we consider the map on cohomology classes.

By use of Theorem 3.5, it only remains to show that

$$\pi^* \circ \Lambda_q \circ [\Lambda_q]^{-1} : H^{n,q}_{(2)}(M, F \otimes [V]^{sing}) \longrightarrow H^{n,q}(M', \pi^* F \otimes [V'])$$

is an isomorphism. This is equivalent to showing that

$$\pi^* \circ \Lambda_q : \check{H}^q(\mathcal{U}, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \to H^{n,q}(M', \pi^* F \otimes [V'])$$

is an isomorphism, where $\mathcal{U} = \{U_\alpha\}$ is a Stein cover for M and Λ_q is the DeRham-Weil-Dolbeault map with respect to a suitable partition of unity $\{\chi_\alpha\}$ subordinate to $\mathcal{U}$. But $\pi^* \circ \Lambda_q = \Lambda'_q \circ \pi^*$, where we let Λ'_q denote the DeRham-Weil-Dolbeault map with respect to the covering $\pi^*\mathcal{U} = \{\pi^{-1}(U_\alpha)\}$ and the partition of unity $\{\pi^*\chi_\alpha\}$ on M'. But

$$\pi^* : \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \xrightarrow{\cong} \pi_*\big(\mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big),$$

by Theorem 3.2, and so:

$$\pi^* : \check{H}^q(\mathcal{U}, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)) \xrightarrow{\cong} \check{H}^q(\pi^*\mathcal{U}, \mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')).$$

On the other hand, Theorem 3.7 tells us that

$$R^q\pi_*\big(\mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big) = 0 \qquad \text{for} \quad q > 0,$$

so that

$$H^q\big(\pi^{-1}(U), \mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')\big) = H^q\big(U, \mathcal{K}_M(F) \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)\big)$$

on open sets $U \subset M$. Hence, $\pi^*\mathcal{U}$ is a Leray cover for $\mathcal{K}_{M'}(\pi^* F) \otimes \mathcal{O}(V')$ on M', meaning that the DeRham-Weil-Dolbeault map Λ'_q on M' is also an isomorphism. Hence, $\pi^* \circ \Lambda_q = \Lambda'_q \circ \pi^*$ is an isomorphism. ∎

The proof of Theorem 4.6 yields immediately:

Corollary 4.7. *The vertical map π^* on the left-hand side of the commutative diagram* (44) *in Theorem 3.5 is surjective.*

4.4. **Extension of L^2-cohomology classes.** The commutative adjunction diagram for L^2-cohomology classes (51) can be understood as a bimeromorphically invariant version of the L^2-extension problem.

From a theorem of Berndtsson, [B2, Theorem 3.1] (see Theorem 1.1), we can now deduce easily the following extension theorem which holds under quite weak positivity assumptions on the Hermitian line bundle $F \to M$, cf. Theorem 1.7.

Theorem 4.8. *Under the assumptions of Theorem 4.6, let $e^{-2\phi}$ be the smooth metric (with weight ϕ) of the Hermitian line bundle $F \to M$. Assume furthermore that M is a Kähler manifold with Kähler metric ω.*

Let $0 \leq q \leq n-1$. Then the adjunction map

$$\Psi_V : H^{n,q}_{(2)}(M, F \otimes [V]^{sing}) \longrightarrow H^{n-1,q}_{(2)}(V^*, F) \tag{52}$$

is surjective if the following condition is satisfied:

There exists an $\epsilon > 0$ such that

$$i\partial\overline{\partial}\phi \wedge \omega^q \geq \epsilon i\partial\overline{\partial}\psi \wedge \omega^q \tag{53}$$

and

$$i\partial\overline{\partial}\phi \wedge \omega^q \geq 0, \tag{54}$$

where $e^{-2\psi}$ is the singular metric on $[V]^{sing}$ which is smooth on $\mathcal{O}(V) \otimes \mathcal{J}(V)$.

Proof. By Theorem 4.6, we can consider instead the extension problem for the smooth hypersurface V' in M'. So, we have to discuss the question whether

$$\Psi_{V'} : H^{n,q}(M', \pi^* F \otimes [V']) \longrightarrow H^{n-1,q}_{(2)}(V', \pi^* F)$$

is surjective, where $\pi^* F$ carries the smooth Hermitian metric $\pi^* e^{-2\phi}$ and $[V']$ carries the smooth Hermitian metric $\pi^* e^{-2\psi}$. As $\pi : M' \to M$ is holomorphic, (53) and (54) give

$$\begin{aligned} \big(i\partial\overline{\partial}\pi^*\phi - \epsilon i\partial\overline{\partial}\pi^*\psi\big) \wedge (\pi^*\omega)^q &\geq 0 \ , &(55)\\ i\partial\overline{\partial}\pi^*\phi \wedge (\pi^*\omega)^q &\geq 0. &(56)\end{aligned}$$

We may assume that the embedded resolution of V in M is obtained by finitely many blow-ups (i.e. monoidal transformations) along smooth centers, see [BM, Theorem 13.4]. So, M' can be interpreted as a submanifold in a finite product of Kähler manifolds and it inherits a Kähler metric ω'.

As ω' is strictly positive definite and $\pi^*\omega$ is only positive semi-definite, there exists a constant $C > 0$ such that $\omega' \geq C\pi^*\omega$. Thus, (55) and (56) imply

$$\begin{aligned} \big(i\partial\overline{\partial}\pi^*\phi - \epsilon i\partial\overline{\partial}\pi^*\psi\big) \wedge (\omega')^q &\geq C^q\big(i\partial\overline{\partial}\pi^*\phi - \epsilon i\partial\overline{\partial}\pi^*\psi\big) \wedge (\pi^*\omega)^q \geq 0 \ ,\\ i\partial\overline{\partial}\pi^*\phi \wedge (\omega')^q &\geq C^q i\partial\overline{\partial}\pi^*\phi \wedge (\pi^*\omega)^q \geq 0.\end{aligned}$$

So, the smooth metrics $\pi^* e^{-2\phi}$ and $\pi^* e^{-2\psi}$ satisfy the assumptions of Berndtsson's extension Theorem 3.1 in [B2] (Theorem 1.1) for the smooth divisor V' in the Kähler manifold (M', ω'), and this gives the required surjectivity of $\Psi_{V'}$. $\square$

Recall that (52) is also surjective if $H^{n,q+1}(M,F)=0$ by (31) and the long exact cohomology sequence (see Section 3.2). That happens by the Kodaira vanishing theorem e.g. if $F\to M$ is a positive line bundle.

Combining Theorem 4.8 with Lemma 4.5, we obtain the following result, cf. Corollary 1.8.

Corollary 4.9. *Under the assumption that Ψ_V is surjective in Theorem 4.8, let $u\in\Gamma\big(V,\mathcal{C}_V^{n-1,q}(F)\big)$ be a $\overline{\partial}$-closed L^2-form of degree $(n-1,q)$ on the singular hypersurface V.*

If $q\geq 1$, then there exists an L^2-form $g\in\Gamma\big(V,\mathcal{C}_V^{n-1,q-1}(F)\big)$ and a $\overline{\partial}$-closed section $U\in\Gamma\big(M,\mathcal{C}^\infty_{n,q}(F)\otimes\mathcal{O}(V)\otimes\mathcal{J}(V)\big)$, i.e. a smooth (n,q)-form with values in $F\otimes[V]$ with some extra vanishing according to $\mathcal{J}(V)$, such that locally

$$U=\frac{df}{f}\wedge(u-\overline{\partial}g)$$

where f is any local defining function for the hypersurface V.

For $q=0$ the statement holds without g.

5. Examples for the multiplier ideal sheaf $\mathcal{J}(V)$

We shall illustrate the role of the multiplier ideal sheaf $\mathcal{J}(V)$ and of our adjunction sheaf $\mathcal{O}(V)\otimes\mathcal{J}(V)$, respectively, in three simple examples.

Example 5.1. Let us discuss shortly what would happen if we blew up a regular hypersurface. So, let V be the regular hypersurface in $\mathbb{C}^2$ (with coordinates z_1,z_2) given as the zero set of $f(z)=z_1$. Let $\pi: M'\to\mathbb{C}^2$ be the blow up of the origin, i.e. M' is given by the equation $z_1w_2=z_2w_1$ in $\mathbb{C}^2\times\mathbb{CP}^1$ with coordinates $((z_1,z_2);[w_1:w_2])$ and π is the projection $\mathbb{C}^2\times\mathbb{CP}^1\to\mathbb{C}^2,(z,w)\mapsto z$.

We cover M' by two charts. The first is given by $w_1=1$ (coordinates z_1,w_2). Here, $\pi^*f=z_1$ and the exceptional divisor E appears as $\{z_1=0\}$.

The second chart is given by $w_2=1$ (coordinates w_1,z_2). Here, $\pi^*f=z_2w_1$, the exceptional divisor E appears as $\{z_2=0\}$, and the strict transform V' of V is just $\{w_1=0\}$.

Thus, in the notation of Section 3.2, we have $E_f=E$ so that E_f is generated by the two holomorphic functions π^*g_1 and π^*g_2 where $g_1=z_1$ and $g_2=z_2$. So, $\mathcal{J}(V)=\mathcal{J}(\varphi)$ with $\varphi=\log\big(|z_1|+|z_2|\big)$. Let $h\in(\mathcal{O}_{\mathbb{C}^2})_0$ be a germ of a holomorphic function at the origin of $\mathbb{C}^2$. Then $he^{-\varphi}=h/(|z_1|+|z_2|)$ is locally square-integrable at the origin. We conclude that $\mathcal{J}(V)=\mathcal{J}(\varphi)=\mathcal{O}_{\mathbb{C}^2}$, which is expected since V is smooth.

Example 5.2. Let V be the cusp in $\mathbb{C}^2$ (with coordinates z_1,z_2) given as the zero set of $f(z)=z_1^3-z_2^2$. We obtain an embedded resolution $\pi:(V',M')\to(V,\mathbb{C}^2)$ with only normal crossings by a sequence of three blow-ups. M' can be realized as follows. We consider $\mathbb{C}^2\times\mathbb{CP}^1\times\mathbb{CP}^1\times\mathbb{CP}^1$ with coordinates $\big((z_1,z_2);[w_1:w_2];[x_1:x_2];[y_1:y_2]\big)$ and define M' by the three equations

$$z_1w_2=z_2w_1,\ \ z_1x_2=w_2x_1,\ \ x_1y_2=w_2y_1.$$

The resolution π is given by the projection on the first factor $\mathbb{C}^2$. The exceptional set consists of three copies E_1,E_2,E_3 of $\mathbb{CP}^1$ coming from the three blow-ups. It is not hard to check that $(\pi^*f)=V'+2E_1+3E_2+6E_3$. The whole resolution M' can be covered by eight charts, but we can get a good picture by just considering two of them.

The first is given by $w_1 = x_2 = y_1 = 1$ (coordinates x_1, y_2). Then $z_1 = x_1^2 y_2$ and $z_2 = x_1^3 y_2^2$ so that $\pi^* f = x_1^6 y_2^3 (1 - y_2)$. Here, $E_2 = (y_2)$, $E_3 = (x_1)$ and $V' = \{1 - y_2 = 0\}$.

The second interesting chart is given by $w_1 = x_2 = y_2 = 1$ (coordinates y_1, w_2). Then $z_1 = y_1 w_2^2$ and $z_2 = y_1 w_2^3$ so that $\pi^* f = w_2^6 y_1^2 (y_1 - 1)$. Here, $E_1 = (y_1)$, $E_3 = (w_2)$ and $V' = \{y_1 - 1 = 0\}$.

One can check that $E_f = 2E_1 + 3E_2 + 6E_3$ is generated by the two holomorphic functions $\pi^* g_1$ and $\pi^* g_2$ where $g_1 = z_1^3$ and $g_2 = z_2^2$. So, we obtain here that $\mathcal{J}(V) = \mathcal{J}(\varphi)$ with $\varphi = \log\big(|z_1|^3 + |z_2|^2\big)$. It is a standard exercise to compute the multiplier ideal sheaf $\mathcal{J}(\varphi)$ (see e.g. [D3], Exercise 15.7):

$$\mathcal{J}(V) = \mathcal{J}(\varphi) = (z_1, z_2),$$

i.e. we obtain the ideal sheaf of the origin (with multiplicity one).

It is now interesting to check that this makes sense in view of the adjunction mapping

$$\Psi : \mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) \to \mathcal{K}_V.$$

A germ ω of $\mathcal{K}_{\mathbb{C}^2} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V)$ can be written as

$$\omega = g \frac{dz_1 \wedge dz_2}{f} = g \frac{dz_1 \wedge dz_2}{z_1^3 - z_2^2},$$

where g is a germ of a holomorphic function in $\mathcal{J}(V)$. As a cusp, V has a well-defined tangential space at the origin, that is $T_0 V = \{z_2 = 0\}$. Thus, $|dz_1|_V \sim 1$ in a neighborhood of the origin on V (measured in the metric on V induced by the Euclidean metric of $\mathbb{C}^2$). Recall that

$$\Psi(\omega) = -g \frac{dz_1}{\partial f / \partial z_2} = g \frac{dz_1}{2 z_2}.$$

Recall that by definition $\Psi(\omega)$ is in the Grauert-Riemenschneider canonical sheaf $\mathcal{K}_V$ if and only if it is square-integrable on V. But

$$|\Psi(\omega)|_V = \left| g \frac{dz_1}{2 z_2} \right|_V \sim |g| \frac{1}{|z_1|^{3/2}},$$

which would not be square-integrable on V at the origin if g were just a holomorphic function. But g is of the form $g = z_1 h_1$ or $g = z_2 h_2$ so that $\Psi(\omega)$ is in fact square-integrable on V. This illustrates the role of the multiplier ideal sheaf $\mathcal{J}(V)$ in the adjunction formula.

Example 5.3. Consider the hypersurface V generated by $f(x, y, z) = z^2 - xy$ in $\mathbb{C}^3$ (with coordinates x, y, z). The embedded resolution $\pi : (V', M') \to (V, M)$ is obtained by a single blow-up of the origin. The exceptional set E is a single copy of $\mathbb{CP}^2$ and it easy to check that $E_f = 2E$ as f vanishes to order 2 in the origin.

One can check that $E_f = 2E$ is generated by the three holomorphic functions $\pi^* g_1$, $\pi^* g_2$ and $\pi^* g_3$ where $g_1 = x^2$, $g_2 = y^2$ and $g_3 = z^2$. It follows that $\mathcal{J}(V) = \mathcal{J}(\varphi)$ with $\varphi = \log\big(|x|^2 + |y|^2 + |z|^2\big)$. But if h is a holomorphic function, then $h e^{-\varphi} = h/(|x|^2 + |y|^2 + |z|^2)$ is square-integrable in $\mathbb{C}^3$. Thus, $\mathcal{J}(V) = \mathcal{J}(\varphi) = \mathcal{O}_{\mathbb{C}^3}$, i.e. V has a canonical singularity (see Theorem 1.4).

We shall check that the adjunction map

$$\Psi : \mathcal{K}_{\mathbb{C}^3} \otimes \mathcal{O}(V) \otimes \mathcal{J}(V) = \mathcal{K}_{\mathbb{C}^3} \otimes \mathcal{O}(V) \longrightarrow \mathcal{K}_V$$

makes sense. A section ω of $\mathcal{K}_{\mathbb{C}^3} \otimes \mathcal{O}(V)$ can be written as

$$\omega = g\frac{dx \wedge dy \wedge dz}{z^2 - xy},$$

where g is just a holomorphic function. To check that $\Psi(\omega)$ is square-integrable on V, we cover the variety V with two charts. For that purpose we cover V by the two parts where either $|x| \geq |y|$ or $|y| \geq |x|$, respectively.

Let $|x| \geq |y|$. Note that on V, this is equivalent to $|z| \leq |x|$. So, we can represent V as a graph with bounded gradient over $\mathbb{C}^2$ with coordinates x, z under the map $y = G(x,z) = z^2/x$. For the gradient, we get $\nabla G = (-z^2/x^2, 2z/x)$, which is in fact bounded as $|z| \leq |x|$. In the coordinates x, z we have

$$\Psi(\omega) = -g\frac{dx \wedge dz}{\partial f/\partial y} = g\frac{dx \wedge dz}{x}$$

so that

$$|\Psi(\omega)|_V \lesssim \frac{1}{|x|}.$$

But this is in fact square-integrable in $\mathbb{C}^2$ with coordinates x, z over the region $|z| \leq |x|$ where we have to integrate (the pole of $1/x$ is only met in $0 \in \mathbb{C}^2$).

The second chart is completely analogous by symmetry. Let $|y| \geq |x|$. On V, this is equivalent to $|xy| \leq |y|^2$ or $|z| \leq |y|$. So, we can represent V as a graph with bounded slope over $\mathbb{C}^2$ with coordinates y, z under the map $x = G(y,z) = z^2/y$. In the coordinates y, z we have

$$\Psi(\omega) = g\frac{dy \wedge dz}{\partial f/\partial x} = -g\frac{dy \wedge dz}{y}$$

so that

$$|\Psi(\omega)|_V \sim \frac{1}{|y|}$$

is square-integrable over the region $|z| \leq |y|$ in $\mathbb{C}^2$.

That shows that $\Psi(\omega)$ is in fact square-integrable over V, thus a section in $\mathcal{K}_V$.

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